si p prod. ex tribus inaequalibus	abc erit	$-\frac{1}{p^n}$
	aab	$+\frac{0}{p^n}$
	aaa	$+\frac{0}{p^n}$
si p prod. ex quatuor inaequalibus	$abcd$	$+\frac{1}{p^n}$
	$aabc$	$+\frac{0}{p^n}$
	$aabb$	$+\frac{0}{p^n}$
	a^3b	$+\frac{0}{p^n}$

LETTRE XXVIII.

EULER à GOLDBACH.

SOMMAIRE. Suite des recherches précédentes.

d. 26 Novembr. 1739.

Considerans rationem, quae intercedit inter summam seriei $\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} +$ etc. et hanc expressionem

$$(P-1)^2 + 1 - \frac{2}{a\pi^n},$$

deprehendi seriem aliquanto esse minorem ac fore

$$(P-1)^2 + 1 - \frac{2}{a\pi^n} = \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \text{etc.}$$

$+ 2$. summa factorum ex ternis
$- 2$. summa factorum ex quaternis
$+ 2$. summa factorum ex quinis
— etc.
} terminis inaequalibus

seriei $\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+$ etc. Quod si autem duplices istae factorum ex ternis, quaternis etc. summae, quippe quae per inventa Tua habentur, substituantur, prodit aequatio identica; quod idem non dubito, quin interim ipse observaveris, V. C.

Incidi heri in hanc seriem non parum curiosam

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{2}{4^n}+\frac{1}{5^n}+\frac{2}{6^n}+\frac{1}{7^n}+\frac{3}{8^n}+\frac{2}{9^n}+\frac{2}{10^n}+\frac{1}{11^n}+\frac{4}{12^n}+\text{etc.}$$

cujus numeratores indicant, quot modis denominatores respondentes sint hujus seriei $2^n+3^n+4^n+5^n+$ etc., vel termini ipsi, vel producta ex binis, vel ternis, vel quaternis, vel ita porro. Sic denominator 60^n numeratorem habebit 11, quia 60 his undecim modis componitur:

I. 60.	V. 5. 12.	IX. 2. 5. 6.
II. 2. 30.	VI. 6. 10.	X. 3. 4. 5.
III. 3. 20.	VII. 2. 2. 15.	XI. 2. 2. 3. 5.
IV. 4. 15.	VIII. 2. 3. 10.	

Hujus seriei summam casu, quo $n=2$, inveni esse $=2$; atque initio arbitratus sum, etiam reliquis casibus summam rationaliter exhiberi posse. Verum rem diligentius scrutatus inveni casu $n=4$ summam fore $=\frac{8e^{\pi}\pi}{e^{2\pi}-1}=\frac{8\pi}{e^{\pi}-e^{-\pi}}$, ubi est proxime $e^{\pi}=23{,}1407$.

Deinde omnia fere theoremata, quae de seriebus numerorum primorum aliisque hinc natis protulisti, V. C., multo latius patere observavi. Si enim sit

$$A=\alpha=a+b+c+d+\text{etc.}$$

$B=$ summae factorum ex binis
$C=$ „ „ ex ternis
$D=$ „ „ ex quaternis
etc.
} terminis seriei A, terminis aequalibus non exceptis,

itemque

$\beta=$ summae factorum ex binis
$\gamma=$ „ „ ex ternis
$\delta=$ „ „ ex quaternis
etc.
} terminis inaequalibus seriei A vel α.

fueritque

$$\left.\begin{aligned}1+A+B+C+D+E+\text{etc.}&=s\\ 1-A+B-C+D-E+\text{etc.}&=t\end{aligned}\right\}\text{erit}\left\{\begin{aligned}1+\alpha+\beta+\gamma+\delta+\text{etc.}&=\frac{1}{t}\\ 1-\alpha+\beta-\gamma+\delta-\text{etc.}&=\frac{1}{s}\end{aligned}\right.$$

hincque

$$1+B+D+F+\text{etc.}=\frac{s+t}{2}\qquad 1+\beta+\delta+\zeta+\text{etc.}=\frac{s+t}{2st}$$

$$A+C+E+G+\text{etc.}=\frac{s-t}{2}\qquad \alpha+\gamma+\varepsilon+\eta+\text{etc.}=\frac{s-t}{2st}$$

item

$$(B-\beta)+(C-\gamma)+(D-\delta)+\text{etc.}=s-\frac{1}{t}$$

$$(B-\beta)-(C-\gamma)+(D-\delta)-\text{etc.}=t-\frac{1}{s}$$

$$(C-\gamma)+(E-\varepsilon)+\text{etc.}=\frac{1}{2}(s-t)\left(1-\frac{1}{st}\right)$$

$$(B-\beta)+(D-\delta)+\text{etc.}=\frac{1}{2}(s+t)\left(1-\frac{1}{st}\right)$$

Quod si autem loco terminorum a, b, c, d, etc. sumantur eorum quadrata sitque $A''=\alpha''=a^2+b^2+c^2+d^2+$ etc., hincque series B'', C'', D'', etc., itemque β'', γ'', δ'', etc. simili modo formentur, quo supra B, C, D, etc. β, γ, δ, etc., ex serie $A=\alpha$ fiet:

$$1+A''+B''+C''+D''+\text{etc.}=st$$

et

$$1-\alpha''+\beta''-\gamma''+\delta''-\text{etc.}=\frac{1}{st}$$

unde erit generaliter

$$1 - A + B - C + D - \text{etc.} = \frac{1 + A'' + B'' + C'' + D'' + \text{etc.}}{1 + A + B + C + D + \text{etc.}}$$

atque

$$(1 + \alpha + \beta + \gamma + \text{etc.})\,(1 \quad \alpha + \beta - \gamma + \text{etc.}) =$$
$$1 - \alpha'' + \beta'' - \gamma'' + \delta'' - \text{etc.}$$

Ex his nunc, si pro serie $a + b + c + d + \text{etc.}$ substituatur haec $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}$ secundum numeros primos procedens, sequentur omnia omnino theoremata, quae mecum communicare voluisti. Vale, V. C., ac favere perge Tui observantissimo

L. Eulero.

LETTRE XXIX.

Euler à Goldbach.

Sommaire. Application du calcul intégral à la sommation des séries.

(Sans date.)

Seriei, cujus terminus generalis est $\frac{1}{64x^2 - 64x + 15}$, vel $\frac{1}{2}\left(\frac{1}{8x-5} - \frac{1}{8x-3}\right)$ summa est $= \frac{1}{2}\int\frac{(zz - z^4)\,dz}{1 - z^8} = \frac{1}{2}\int\frac{zz\,dz}{(1+zz)(1+z^4)} = -\frac{1}{4}\int\frac{dz}{1+zz} + \frac{1}{4}\int\frac{(1+zz)\,dz}{1+z^4}$, si post integrationem ponatur $z = 1$. At seriei, cujus terminus generalis est $= \frac{3m}{64xx - 64x + 7} = \frac{m}{2}\left(\frac{1}{8x-7} - \frac{1}{8x-1}\right)$, summa est $= \frac{m}{2}\int\frac{(1-z^6)\,dz}{1-z^8} = \frac{m}{2}\int\frac{(1+zz+z^4)\,dz}{(1+zz)(1+z^4)} = \frac{m}{4}\int\frac{dz}{1+zz} + \frac{m}{4}\int\frac{(1+zz)\,dz}{1+z^4}$, posito post integrationem $z = 1$. Verum est $\int\frac{dz}{1+zz} = \frac{\pi}{4}$; $\int\frac{dz}{1+z^4} = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\,l(1 + \sqrt{2})$ et