How Euler Did It

by Ed Sandifer

Estimating the Basel Problem

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In the lives of famous people, we can often identify the first thing they did that made them famous. For Thomas Edison, it was probably his invention of the phonograph in 1877. Abraham Lincoln first made his name in the Lincoln-Douglas Debates of 1858, though Steven A. Douglas won that election for the U. S. Senate, not Abraham Lincoln.

Leonhard Euler's first celebrated achievement was his solution in 1735 of the "Basel Problem", finding an exact value of the sum of the squares of the reciprocals of the integers, that is $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots$. Bill Dunham [D] gives a wonderful account of Euler's solution in his book *Euler The Master of Us All*, published by the MAA in 1999. However, just as Edison's invention of the phonograph depended critically on his invention of waxed paper a few years earlier, so also Euler's solution to the Basel Problem had its roots in a result from 1730 on estimating integrals.

Pietro Mengoli (1625-1686) posed the Basel problem in 1644. The problem became well known when Jakob Bernoulli wrote about it in 1689. Jakob was the brother of Euler's teacher and mentor Johann Bernoulli, who probably showed the problem to Euler. By the 1730's, the problem had thwarted many of the day's best mathematicians, and it had achieved the same kind of mystique that Fermat's Last Theorem had before 1993.

In 1730 Euler is interested in problems he calls "interpolation of series." Given a process defined for whole numbers, he seeks meaningful ways to extend the definitions of those processes to non-integer values. For example, he already extended what he called the hypergeometric series and we call the factorial function, $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ to give a definition that worked for fractional values. The function he devised is now called the Gamma function.

In the paper that bears the index number E20, Euler does the same thing for the partial sums of the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$. If the first partial sum of this series is 1 and the second is $\frac{3}{2}$ and the third is $\frac{11}{6}$, and so forth, Euler asks what value might be assigned to the $\frac{1}{2}th$ or the $\frac{3}{2}th$ partial sums. Euler's answer lies in integration and geometric series.

First, Euler recalls the formula for the sum of a finite geometric series,

$$1 + x + x^2 + x^3 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

Here, n is the number of terms, and the formula can be applied for any value of n, even though it may not be clear what the formula might mean if n is not a whole number.

Now, Euler integrates both sides. On the left, he gets $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots + \dfrac{x^n}{n}$. Taking $x = 1$ gives the

desired nth partial sum of the harmonic series. On the right, he gets $\displaystyle\int \dfrac{1-x^n}{1-x}\,dx$. Our modern notation for definite

integrals had not yet evolved, so instead of writing $\displaystyle\int_0^1 \dfrac{1-x^n}{1-x}\,dx$ Euler adds that we should "take it $= 0$ if $x = 0$, and then set x

$= 1$." So Euler uses this integral as the value for the nth partial sum of the harmonic series, even if n is not a whole number.

For most values of n, the integral $\displaystyle\int_0^1 \dfrac{1-x^n}{1-x}\,dx$ is rather difficult to evaluate. In the case $n = \dfrac{1}{2}$, though, it is not

too difficult to find that $\displaystyle\int_0^1 \dfrac{1-\sqrt{x}}{1-x}\,dx = \int_0^1 \dfrac{1}{1+\sqrt{x}}\,dx = 2 - 2\ln 2$, thus giving a value to the $\dfrac{1}{2}th$ partial sum.

    Euler can always find more to do with a new technique. This integration idea works because integration gives a relation between geometric and harmonic series. In particular, (overlooking, as Euler did, the constant of integration)

$$\int \frac{1-x^n}{1-x}\,dx = \int \left(1 + x + x^2 + \ldots + x^{n-1}\right)dx = x + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^n}{n}$$

If we integrate again, then on the right hand side we get

$$\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \ldots + \frac{x^{n+1}}{n\cdot(n+1)}.$$

Here, the denominators are products of consecutive integers. These are tantalizingly like, but are not exactly squares. The Basel Problem seems to be lurking nearby.

    Euler, though, sees a way to get the perfect squares in the denominators that he wants so badly. He divides by x before integrating again. He sees that

$$\int \frac{1}{x}\left(x + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^n}{n}\right)dx = x + \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 3} + \ldots + \frac{x^n}{n\cdot n},$$

again, neglecting the constant of integration. Euler doesn't have a very good notation for double integrals, either, so he writes this in a way that we would now regard as nonsense:

$$\int \frac{dx}{x} \int \frac{1-x^n}{1-x}\,dx = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}$$

Euler does give us instructions about how to evaluate this, and they translate into modern notation to be

$$\int_0^1 \frac{1}{x}\left(\int_0^x \frac{1-y^n}{1-y}\,dy\right)dx$$

To solve the Basel Problem all Euler has to do is evaluate this integral and then take its limit as n goes to infinity. This does not seem like progress, because this integral is very difficult to evaluate. However, Euler uses a very clever approximation technique to get a numerical value, 1.644924, for this integral. The series itself converges rather slowly, and you would have to sum more than 30,000 terms to get this degree of accuracy directly.

    This value does not mean much to most people, but among his other skills, Euler is a fantastic calculator. He sees

this value is close to $\dfrac{p^2}{6}$. Armed with this hint, he begins to attack the Basel Problem using series for trigonometric

functions, in particular, the Taylor Series for $\dfrac{\sin x}{x}$. In fact, Euler was the first one to call them "Taylor's series." Five

years later, he publishes his solution and the world of mathematics knows his name.

[D]   Dunham, William, *Euler The Master of Us All*, MAA, Washington, D. C., 1999.

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