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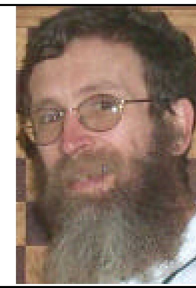
How Euler Did It

by Ed Sandifer



Basel Problem with Integrals

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Euler's brilliant 1734 solution to the Basel problem, to find the value of the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.$ [E 41] brought him great fame, but it also depended on some assumptions that are rather difficult to justify. In particular, at a key point in the solution, Euler notes that the function $\frac{\sin x}{x}$ and the infinite product $\left(1+\frac{x}{p}\right)\left(1-\frac{x}{p}\right)\left(1+\frac{x}{2p}\right)\left(1-\frac{x}{2p}\right)\left(1+\frac{x}{3p}\right)\left(1-\frac{x}{3p}\right)\ldots$ have exactly the same roots and have the same value at x = 0, so Euler asserts that they describe the same function. Euler is correct that they describe the same function, but these reasons are insufficient to guarantee it. For example, the function $e^x\frac{\sin x}{x}$ also has the same roots and the same value at x = 0, but it is a different function. This seems to be a modern objection, not raised in Euler's time.

Nonetheless, Euler seemed to understand that there was something mysterious or incomplete in his explanation of this step. He wrote some other papers, for example E 61, in which he tried to extend and justify this infinite product technique, but he never got very far with clearing the fog out of the solution.

It is generally assumed that was where Euler left the issue. However, in 1741, he wrote a seldom-read paper in French, published in a rather obscure literary journal in which he gives a completely different solution to the Basel problem, one that does not depend on the mysteries of infinite products.

Euler begins by asking us to consider a circle of radius 1. He takes s to be arc length, and takes $x = \sin s$, or, equivalently, $s = \arcsin x$. Then, working with differentials as he always does, $ds = \frac{dx}{\sqrt{1-xx}}$ and $s = \int \frac{dx}{\sqrt{1-xx}}$. Now, he multiplies these together to get $sds = \frac{dx}{\sqrt{1-xx}}\int\frac{dx}{\sqrt{1-xx}}$. He integrates both sides from $x=0$ to $x=1$. On the left, the antiderivative is $\frac{ss}{2}$, and, as x goes from 0 to 1, s goes from 0 to $\frac{p}{2}$, so he gets on the left $\frac{pp}{8}$.

On the right, Euler dives fearlessly into an intricate series calculation. He writes $\frac{1}{\sqrt{1-xx}} = (1-xx)^{\frac{-1}{2}}$, and applied the generalized binomial theorem to expand the radical as an infinite series. He gets

$$(1-xx)^{\frac{-1}{2}} = 1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8+etc.$$

This is a bit of a tricky step, but it really is the familiar binomial theorem, that

$$(1+a)^n = 1+\frac{n}{1}a+\frac{n(n-1)}{1\cdot 2}a^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3+\ldots$$

in the case $a = xx$ and $n = \frac{-1}{2}$. If n is a positive integer, then the numerators eventually become zero, and we get a finite sum, but the theorem is still true if n is a fraction. Euler's series converges whenever $|x|<1$. He integrates and multiplies to get that

$$sds = \frac{xdx}{\sqrt{1-xx}}+\frac{1}{2\cdot 3}\frac{x^3\,dx}{\sqrt{1-xx}}+\frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{x^5\,dx}{\sqrt{1-xx}}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{x^7\,dx}{\sqrt{1-xx}}+\ldots$$

He knows that if he integrates both sides of this, from $x = 0$ to $x = 1$, he will get $\frac{pp}{8}$. If he integrates an individual term on the right, using integration by parts, he gets

$$\int \frac{x^{n+2}dx}{\sqrt{1-xx}} = \frac{n+1}{n+2}\int \frac{x^n dx}{\sqrt{1-xx}} - \frac{x^{n+1}}{n+2}\sqrt{1-xx}$$

Since the second term is zero at both endpoints, he can ignore it, and he gets a nice reduction formula. He summarizes the integral result with a list:

$$\int_0^1 \frac{xdx}{\sqrt{1-xx}} = 1$$

$$\int_0^1 \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2}{3}\int_0^1 \frac{xdx}{\sqrt{1-xx}} = \frac{2}{3}$$

$$\int_0^1 \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{4}{5}\int_0^1 \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5}$$

$$\int_0^1 \frac{x^7\,dx}{\sqrt{1-xx}} = \frac{6}{7}\int_0^1 \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

So, the integral of the expression above (the one that begins $s\,ds$) is

$$\frac{pp}{8} = \int_0^1 \frac{xdx}{\sqrt{1-xx}}+\frac{1}{2\cdot 3}\int_0^1 \frac{x^3\,dx}{\sqrt{1-xx}}+\frac{1\cdot 3}{2\cdot 4\cdot 5}\int_0^1 \frac{x^5\,dx}{\sqrt{1-xx}}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\int_0^1 \frac{x^7\,dx}{\sqrt{1-xx}}+etc.$$

Substituting the values for the integrals gives

$$\frac{pp}{8} = 1+\frac{1}{3\cdot 3}+\frac{1}{5\cdot 5}+\frac{1}{7\cdot 7}+\frac{1}{9\cdot 9}+etc.$$

The series on the right is the sum of the reciprocal of the odd squares, tantalizingly close to the Basel problem, and an easy trick makes it into a solution. Any number is the product of an odd number and a power of 2. For odd numbers, the power of 2 is 2^0. Hence, any square is the product of an odd square and a power of 4. So, Euler multiplies this equation by the sum of the reciprocals of the powers of 4, as follows:

$$\frac{4}{3} = 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$$

and gets

$$\frac{pp}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots$$

It is a different solution to the Basel problem that does not depend on infinite products. In fact, all it requires to meet modern standards of rigor is that we fill in a few routine steps and notice that a few series are absolutely convergent, so that we can do things like multiply two different series together, as we did in the very last step.

References

Euler, L., "De summis serierum reciprocarum", *Opera Omnia* I.14, pp. 73-86, (E 41)

Euler, L., "Demonstration de la somme de cette suite $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + etc.$", *Opera Omnia* I.14, pp. 177-186, (E 63).

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How Euler Did It is updated each month.