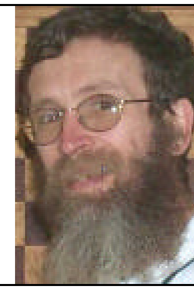
How Euler Did It

by Ed Sandifer



Theorema Arithmeticum

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Euler's 1748 textbook, the *Introductio in analysin infinitorum*, was one of the most influential mathematics books of all time. John Blanton's excellent translation, is available in many libraries, and on the bookshelves of those few individuals who were lucky enough to get copies in the relatively brief time it was in print. Those people seem to be holding on to their copies, as there seem to be almost none available on the used book market.

If you read the *Introductio*, you are likely to have different reactions to different parts of the book. When you read the section about partial fractions or the section about the definition the trigonometric functions, you will feel very much at home. Euler's treatment is very similar to the way we present these topics today. This is because, for this topic, how Euler did it became adopted as the standard way to do it.

When you read about series, you may feel like Euler is doing some things the hard way because he doesn't use calculus. This is a precalculus book, an *introduction* to the methods and material that will be used in calculus, so he does not use Taylor series or other calculus tools. It is surprising how much he is able to do without calculus.

Some other topics, like partitions and continued fractions, aren't seen so often any more, and it is exciting to see how much can be done by elementary means.

The *Introductio* provides a kind of foundation for much of Euler's career. Time and time again he finds a lemma in the *Introductio* that he needs in some later paper, or he writes a whole paper that begins with a topic from the *Introductio*.

This month's column, though, isn't about the *Introductio*. It is about something that would have fit well with some of the other material in the *Introductio*. Maybe it should have been there. When Euler needed this little result, it wasn't there, so he had to pause to prove it.

Euler wrote a massive text on calculus. His *Institutiones calculi differentialis*, E-212, was published in 1755, seven years after the *Introductio*. More than ten years later, in 1769, E-342, E-366 and E-385, his three volumes of the *Institutiones calculi integralis* came out. At more than 2500 pages, these four volumes outweigh even the most prolix of modern texts. Though Euler was seldom accused of being too brief, we should deflect some criticism of his verbiage; he does include both an extensive treatment of differential equations and a good bit of the calculus of variations under his umbrella of "calculus."

The whole calculus series is presented as a series of problems. Book I of the *Calculis integralis*, for example, has 173 problems, spread across two volumes. Each problem is given in rather general form, and with a general solution. Most solutions are followed by a number of corollaries, scholions or examples. Each problem, corollary or other part has a "paragraph" number, though most consist of more than one paragraph. Book I has 1275 such paragraphs.

Our example comes from near the end of Book I of the *Calculis integralis*, part of volume 2 of book I, so this is found in E-366. It follows Problem 152 and is in paragraph 1169. At this point, Euler has been doing differential equations for over 300 pages. He comes upon a rather complicated problem (there isn't space to get in to it here) that can be dramatically simplified using a clever partial fraction expansion.

Normally at this point, Euler would refer to the *Introductio* to find the lemma that solves the problem. This time, the result isn't there! So Euler pauses to give us:

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Given numbers *a, b, c, d,* etc., if from each one is subtracted each other one and the following products are formed:

$$(a-b)(a-c)(a-d)(a-e) \text{ etc.} = \mathfrak{a}$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc.} = \mathfrak{b}$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc.} = \mathfrak{g}$$
$$(d-a)(d-b)(d-c)(d-e) \text{ etc.} = \mathfrak{d}$$

then it will always be that

$$\frac{1}{\mathfrak{a}}+\frac{1}{\mathfrak{b}}+\frac{1}{\mathfrak{g}}+\frac{1}{\mathfrak{d}}+ \text{ etc.} = 0.$$

Euler overlooks the condition that the numbers *a, b, c, d,* etc., ought to be distinct, or else two of the products will be zero and the formula in the conclusion will be undefined.

If we have three numbers, *a, b* and *c*, then Euler is claiming that

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}=0$$

With a bit of algebra, the reader who is careful with signs can easily verify this identity by using $(a-b)(a-c)(b-c)$ as a common denominator.

The case of four numbers, though, would require a common denominator with six factors, $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$, and the algebra is considerably more cumbersome. In general, *m* numbers would require $\binom{m}{2} = \frac{m^2-m}{2}$ factors. This quickly moves from the awkward to the infeasible. There must be a better way.

The modern reader would probably rewrite Euler's claim using subscripts, sigmas and product symbols. Let the *m* numbers be $a_1,\ a_2,\ a_3,\ldots,a_m$, and the *m* products be given by $\mathfrak{a}_i = \prod_{\substack{j=1\\ j\neq i}}^{m}(a_i - a_j)$.

Then Euler claims that $\sum_{i=1}^{m}\frac{1}{\mathfrak{a}_i}=0$. Then, with a careful management of subscripts and symbols, it is probably possible to prove the result. It would probably not seem clever.

Euler though, did not have those tools, so he had to find a clever way.

Euler begins his proof with a step that makes the reader expect a proof by mathematical induction. That's not what he's doing, though. He supposes that the last of his m numbers is denoted by z, and that Z is a polynomial in z of degree less than $m - 1$. He forms the rational expression

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\text{ etc.}}.$$

Note that the denominator here will be the last of the products Euler defined in the statement of his theorem, so that

$$z = (z-a)(z-b)(z-c)(z-d)\text{etc.}$$

Now, since he knows his *Introductio*, he decomposes this into its partial fractions:

$$(*) \qquad \frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}+\frac{D}{z-d}+\text{etc.}$$

We will need the negative of this expression, so Euler notes that its negative is

$$(**) \qquad \frac{A}{a-z}+\frac{B}{b-z}+\frac{C}{c-z}+\frac{D}{d-z}+\text{etc.}$$

He is really only interested in the special case where $Z = z^n$, where n is less than $m - 1$. For this particular Z, we can do just a little work, we can find the numbers A, B, C, etc. explicitly as

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\text{etc.}}$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\text{etc.}}$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\text{etc.}}$$

etc.

The last factors of these denominators are $(a - z)$, $(b - z)$, $(c - z)$, etc. Since Euler will be interested in the expansion of $\frac{z^n}{z}$, he will be using fractions for which the denominators involve $(z - a)$, $(z - b)$, $(z - c)$, etc., hence his remark that gave us the equation marked (**).

With this groundwork set out, Euler is ready to look at the problem itself. Taking y to denote the penultimate term, the products given in the theorem are now:

$$
\begin{aligned}
&(a-b)(a-c)(a-d)\ldots(a-z)=\mathbf{a}\\
&(b-a)(b-c)(b-d)\ldots(b-z)=\mathbf{b}\\
&(c-a)(c-b)(c-d)\ldots(c-z)=\mathbf{g}\\
&(d-a)(d-b)(d-c)\ldots(d-z)=\mathbf{d}\\
&\text{etc.}\\
&(z-a)(z-b)(z-c)\ldots(z-y)=\mathbf{z}
\end{aligned}
$$

We notice that

$$\frac{z^n}{\mathbf{z}}=\frac{z^n}{(z-a)(z-b)(z-c)\ldots(z-y)}.$$

Keep in mind that we just did a partial fraction expansion of this as we do a couple of preliminary calculations. We see that

$$
\begin{aligned}
\frac{a^n}{\mathbf{a}}&=\frac{a^n}{(a-b)(a-c)\ldots(a-y)(a-z)}\\
&=\frac{a^n}{(a-b)(a-c)\ldots(a-y)}\cdot\frac{1}{(a-z)}\\
&=A\cdot\frac{1}{a-z}\\
&=\frac{-A}{z-a}
\end{aligned}
$$

Similarly for $\frac{b^n}{\mathbf{b}}$, $\frac{c^n}{\mathbf{g}}$, etc. Now, putting this together with our partial fraction expansion, we can do the following calculation:

$$
\begin{aligned}
\frac{z^n}{\mathbf{z}}&=\frac{z^n}{(z-a)(z-b)(z-c)\ldots(z-y)}\\
&=\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}+\ldots+\frac{Y}{z-y}\\
&=-\frac{a^n}{\mathbf{a}}-\frac{b^n}{\mathbf{b}}-\frac{c^n}{\mathbf{g}}-\ldots-\frac{y^n}{\mathbf{u}}
\end{aligned}
$$

Now comes the punch line. From this last equation we get

$$\frac{a^n}{\mathbf{a}}+\frac{b^n}{\mathbf{b}}+\frac{c^n}{\mathbf{g}}+\ldots+\frac{z^n}{\mathbf{z}}=0.$$

Taking $n = 0$ gives the desired result.

There are probably many other ways to prove this result, but probably no other way has such an unexpected and surprise ending, and still uses only 18th century methods. Once again, Euler shows why he was the greatest of his century.

References:

[E101] Euler, Leonhard, *Introduction to the Analysis of the Infinite*, translated by John D. Blanton, Springer, New York, 1988 (v. 1), 1990 (v. 2)

[E366] Euler, Leonhard, *Institutiones calculi integralis, volumen secundum*, St. Petersburg, 1769, reprinted in *Opera Omnia* Series I vol 12.

Thanks to Rob Bradley for his help with this column.

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A new issue of *How Euler Did It* appears each month.