How Euler Did It

by Ed Sandifer

Euler and Pell

April, 2005

Much of a modern course in elementary number theory has its roots in Euler (though the notation is largely due to Gauss.) Euler, in turn, cites as his inspiration the works of Fermat, Diophantus, Goldbach and Pell, among others. This month we will look at the so-called Pell's Equation, $y^2 = ax^2 + 1$, a, x and y integers, named after the English mathematician John Pell (1611-1685), who lived about a hundred years earlier than Euler.

It was Euler who attached the name "Pell's equation" to this formula. People often say that Euler made a mistake in attributing the equation to Pell, but the entry on Pell at MacTutor [O'C] reports:

> Pell's equation $y^2 = ax^2 + 1$, where a is a non-square integer, was first studied by Brahmagupta and Bhaskara II. Its complete theory was worked out by Lagrange, not Pell. It is often said that Euler mistakenly attributed Brouncker's work on this equation to Pell. However the equation appears in a book by Rahn which was certainly written with Pell's help: some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation.

Euler's first excursion into Pell's equation was his 1732 paper E-29, bearing a title that translates as "On the solution of problems of Diophantus about integer numbers." The main result of this paper is to show how certain quadratic Diophantine equations can be reduced to the Pell equation. In particular, he shows that if we can find a solution to the Diophantine equation $y^2 = an^2 + bn + c$ and we can find solutions to the Pell equation, $q^2 = ap^2 + 1$, then we can use the solutions to the Pell equation to construct more solutions to the original Diophantine equation. He also shows how to use two solutions to a Pell equation to construct more solutions, and notes that solutions to a Pell equation give good rational approximations for $\sqrt{a}$. When Euler discovers the connection between the Pell equation and continued fractions, most of this becomes obsolete, so we will not dwell on it here.

Euler returns to the Pell equation more than 30 years later with his paper "On the use of a new algorithm in solving the Pell problem," E-323. The Summarium at the beginning of the article announces that the new algorithm will enable us to find easily a solution in the case $a = 61$. This is rather dramatic, since the smallest solution to the equation $p^2 = 61q^2 + 1$ has p a ten-digit number and q a nine digit one. To find such solutions by hand would indeed be arduous.

Euler begins to describe his algorithm with an example, using $a = 13$. We know that $\sqrt{13}$ is between 3 and 4, so he writes

$$\sqrt{13} = 3 + \frac{1}{a}$$

where we know that $a > 1$. A bit of algebra finds a to be exactly

$$a = \frac{\sqrt{13} + 3}{4}.$$

Knowing that $3 < \sqrt{13} < 4$ makes it easy to show that $1 < a < 2$, so we write

$$a = \frac{\sqrt{13} + 3}{4} = 1 + \frac{1}{b}$$

where, again, $b > 1$. An almost identical calculation shows that

$$b = \frac{4}{\sqrt{13} - 1} = 1 + \frac{1}{c}.$$

If we pause to take stock of what has happened, we get a clue to what Euler is doing here. If we make the substitutions, we get that

$$\sqrt{13} = 3 + \frac{1}{a}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{b}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{c}}}$$

Euler is building a continued fraction. He continues, finding, in turn, d, e, and f, and he finds that $f = a$, so the process is cyclic, and, after the initial 3, the coefficients that repeat are 1, 1, 1, 1 and 6. He does some more examples, including $a = 61, 67, 31, 46$, then 54. In each case, he notes that there is an initial integer followed by a pattern that repeats. The initial integer, he denotes v, is the integer part of $\sqrt{a}$. This is followed by a palindromic sequence of integers, followed by the integer $2v$. Then the palindrome and the $2v$ repeat. He gives a table of the cycles for all non-square integers from 2 to 120. He also notes some of the very interesting patterns within these cycles. Hardly any of these properties were in E-71, Euler's pioneering paper on continued fractions.

With the existence of these patterns in place, though, he is ready to use some results of his earlier paper. That paper has been translated into English and published in the journal *Mathematical Systems Theory*. [E71] Euler reviews some of his results on how to evaluate continued fractions if you know the pattern of the "indices."

To evaluate the continued fraction corresponding to a sequence of indices, make a table as below:

Indices		v	a	b	c	...	m	n	
x	1	v	$av+1$	$(av+1)b+v$			M	N	$nN+M$
y	0	1	a	$ab+1$			P	Q	$nQ+P$

Today we would say that the sequence of numerators, x and of denominators y, each satisfy a recursive relation of order 2, with initial conditions 1, v and 0, 1 respectively.

Let's do an example. The indices for $\sqrt{3}$ are 1, 1, 2, 1, 2, 1, 2, etc., so $v = 1$, $2v = 2$, and the palindrome is just 1. To evaluate the first several values of the continued fraction

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \text{etc.}}}}}$$

we start with a table

Indices		1	1	2	1	2	1	2
x	$\underline{1}$	$\underline{v}$	$\dfrac{x_2}{}$					
y	0	1	y_2					

Now, $v = 1$ since $1 < \sqrt{3} < 2$. The next index is 1, so $x_2 = 1 \cdot v + 1 = 2$ and $y_2 = 1 \cdot 1 + 0 = 1$, giving

Indices		1	1	2	1	2	1	2
x	$\underline{1}$	$\underline{1}$	$\underline{2}$					
y	0	1	1					

The next index is 2, so we get

Indices		1	1	2	1	2	1	2
x	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{5}$				
y	0	1	1	3				

Continuing,

Indices		1	1	2	1	2	1	2
x	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{5}$	$\underline{7}$	$\underline{19}$	$\underline{26}$	$\underline{71}$
y	0	1	1	3	4	11	15	41

These quotients give progressively better approximations of $\sqrt{3}$, alternating between being too large and being too small. The last one, 71/41, is accurate to three decimal places. But solutions the Pell equation $3qq + 1 = pp$ also have quotients that approximate $\sqrt{3}$. In fact, several, but not all of these quotients give rise to solutions to the equation, $(p, q) = (0, 1)$, $(1, 2)$, $(4,7)$, and $(15, 26)$. In this particular case, the pattern of solutions and non-solutions is fairly simple, but Euler's paper gives rules for the pattern of solutions in every case. We leave finding and describing these patterns to interested readers and to students in search of a number theory project.

Also, some quotients give rise to solutions to $3qq - 1 = pp$, and Euler gives ways use these solutions to find solutions to Pell's equation.

Eight years later, in 1773, Euler returns to the Pell equation with E-559, "New aids for solving the formula $axx + 1 = yy$," not published until 1783. In this paper, he gives ways to generate solutions of the Pell equation from solutions to related equations, $app - 1 = pp$, $app - 2 = pp$, $app + 2 = pp$ and $app + 4 = pp$. This provides a kind of converse to the main results of E-29, published 50 years earlier.

Euler wrote over 160 papers that the editors of the *Opera Omnia* have classified as "number theory." They fill volumes 2, 3, 4 and 5 of series I, 1bout 1700 pages. The three papers we have looked at here comprise only about 2% of Euler's number theory papers, but they extend from one of his very first papers, E-29, to one of his last, E-559, published in 1783, the year Euler died. Though the Pell equation was a relatively minor aspect of Euler's work, it did hold his interest for his whole life. And it is still interesting today.

References:

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Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, planning to do his 33rd Boston Marathon this month, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.