



## Roots by Recursion

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Euler's great work of 1748, *Introductio in analysin infinitorum*, is one of the world's truly great mathematics books. It was one of the first important books to be based on the concept of function, instead of the older idea of curves. It was also one of the first books specifically designed to help a student bridge the gap between algebra and calculus. As such, it, along with Maria Agnesi's book that appeared the same year, should be considered to be the first precalculus book.

After writing the book, Euler had trouble finding a publisher. The academy in St. Petersburg that published many of his other books was suffering under the political turmoil in Russia at the time, so they did not have the resources to publish it. The Berlin Academy, where Euler was employed from 1741 to 1766, did not publish such books, so Euler found a publisher in Lucerne, Switzerland to produce the book. Euler was not able to journey to Lucerne to check the proofs of the book, so he asked his friend in Lucerne, Gabriel Cramer, to do it for him. This is the same Cramer of Cramer's rule in linear algebra and Cramer's paradox, the subject of our column from August 2004. Their correspondence about the production of the book provides some interesting insights into Euler's thoughts and the reception of his ideas in his own time. A portrait of Cramer is at the right.



The *Introductio* was published in two books, with Eneström numbers 101 and 102. Both have been translated by John Blanton and published in two volumes by Springer-Verlag in 1988 and 1990.

This month's column comes from Chapter 17 of Part I of the *Introductio*. The chapter is titled "On the use of recurrent series in finding the roots of equations."

In the Chapter 13, "On recurrent series," Euler had described some results of DeMoivre on the expansion of rational functions into infinite series. In particular, using Euler's notation, if

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - az - bz^2 - cz^3 - dz^4 - \text{etc.}}$$

is a “proper” rational function, that is, the quotient of polynomials with the degree of the numerator less than the degree of the denominator, then it can be expanded into a power series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

and after some complications in the initial terms, successive coefficients ...  $P, Q, R, S$  and  $T$  would satisfy a recurrence relation that Euler wrote as

$$T = aS + bR + gQ + dP + \text{etc.}$$

The initial complications are described by the following rules:

$$\begin{aligned} A &= a \\ B &= aA + b \\ C &= aB + bA + c \\ D &= aC + bB + gA + dd \end{aligned}$$

These are not hard to check by multiplying the series together and matching terms, but we can make sure we know what we’re doing by working through Euler’s example: to expand

$$\frac{1 - z}{1 - z - 2zz}$$

into a series.

If we set  $\frac{1 - z}{1 - z - 2zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$  and multiply through by the  $1 - z - 2zz$ , we get

$$1 - z = (1 - z - 2zz)(A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.})$$

Matching constant terms we get

$$1 \cdot 1 = A \text{ so } A = 1.$$

Matching linear terms we get

$$-z = 1 \cdot Bz - z \cdot A \text{ so } B = 0.$$

Matching quadratic terms we get

$$0 = 1 \cdot Cz^2 - z \cdot Bz - 2zz \cdot A \text{ so } C = B + 2A.$$

After that, the left hand sides are all zero, and the coefficients on the right satisfy relations that proceed

$$\begin{aligned} D &= C + 2B, \\ E &= D + 2C, \\ F &= E + 2D, \text{ etc.} \end{aligned}$$

The reader will want to continue this calculation and find that the first several terms are 1, 1, 3, 5, 11, 21 and 43.

A reader of the *Introductio* learned all of this in Chapter 13 and is ready to use it in Chapter 17. Let us proceed the way Euler does, and make some simplifying assumptions. Let's suppose we want to find a root of a cubic polynomial  $1 - az - bz^2 - gz^3$ . Euler, following his usual pedagogy, starts with second degree polynomials, then does third degree, and then goes on to polynomials of arbitrary degree. We will do the derivation for third degree, but our example will be second degree.

Let us avoid a few complications and suppose also that the polynomial has three distinct real roots of different magnitudes, and, since the constant term is non-zero, none of the roots are zero.

Now, we know that the polynomial factors somehow as  $(1 - pz)(1 - qz)(1 - rz)$ , but if we knew exactly how it factors, we would know the roots are  $1/p$ ,  $1/q$  and  $1/r$ . Having said this, we know that the reciprocal of the polynomial can be written as a partial fraction:

$$\frac{1}{1 - az - bz^2 - gz^3} = \frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz}$$

Euler has to resort to Fraktur characters because subscripts have not yet been invented. We do not need to know  $A$ ,  $B$ ,  $C$  or  $p$ ,  $q$ ,  $r$  to know that the quotient has such a partial fraction representation.

Now, on the left, we can use the results from Chapter 13 to expand the quotient into a power series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

where  $A = 1$ , and every coefficient after that is given by a simple recurrence relation involving  $\alpha$ ,  $\beta$  and  $\gamma$ . Meanwhile, each of the terms on the right can be expanded into geometric series:

$$\frac{A}{1 - pz} = A + Apz + A(pz)^2 + A(pz)^3 + A(pz)^4 + \text{etc.}$$

$$\frac{B}{1 - qz} = B + Bqz + B(qz)^2 + B(qz)^3 + B(qz)^4 + \text{etc.}$$

$$\frac{C}{1 - rz} = C + Crz + C(rz)^2 + C(rz)^3 + C(rz)^4 + \text{etc.}$$

So, matching terms we get

$$A = A + B + C$$

$$B = Ap + Bq + Cr$$

$$C = Ap^2 + Bq^2 + Cr^2$$

and in general, for larger exponents,

$$M = Ap^m + Bq^m + Cr^m$$

$$N = Ap^{m+1} + Bq^{m+1} + Cr^{m+1}$$

Since we assumed that the roots are different magnitudes, one of  $p$ ,  $q$ ,  $r$  has the largest magnitude. Suppose it is  $p$ . This makes  $1/p$  the smallest root. So, for large enough values of  $m$ , the term involving  $p$  dominates the other two terms, so

$$\frac{N}{M} = \frac{Ap^{m+1} + Bq^{m+1} + Cr^{m+1}}{Ap^m + Bq^m + Cr^m} \approx \frac{Ap^{m+1}}{Ap^m} = p$$

and, since  $p$  is the largest of the three coefficients in the factors, it is the smallest (in magnitude) of the three roots.

Let us try this with Euler's own first example: to find a root of  $xx - 3x - 1 = 0$ . Rather than consider  $\frac{1}{1-3z-zz}$ , Euler decides to look at  $\frac{a+bz}{1-3z-zz}$ , where  $a$  and  $b$  are whatever numbers make the sequence of coefficients begin 1, 2, ...

After that, the recurrence relation will give

$$\begin{aligned} C &= 3B + A \\ D &= 3C + B \\ &\text{etc.} \end{aligned}$$

The sequence thus generated is

$$1, 2, 7, 23, 76, 251, 829, 2738, \&c.$$

Taking the last pair of these, we get

$$\begin{aligned} p &= \frac{829}{2738} \\ 1/p &= \frac{2738}{829} \\ &= 3.3027744 \end{aligned}$$

Finding the root by the quadratic formula gives

$$\begin{aligned} x &= \frac{3 + \sqrt{13}}{2} \\ &= 3.3027756 \end{aligned}$$

The error is in the 6<sup>th</sup> decimal place.

Euler does a number of other examples, all of which are designed so that he has a way to check the results. Example IV is to find the root of  $0 = 8y^3 - 24yy + 8y - 1$ , the root of which is  $1 + \sin 70^\circ$ . Euler finds that the sequence that begins 1, 1, 1 ... gives, as the ratio of its 8<sup>th</sup> and 9<sup>th</sup> terms, the correct answer to seven decimal places. Of course, most of Euler's examples are chosen so that they will work well.

Euler describes some enhancements. If, for example, we want to find the largest root of  $f(z)$ , we can substitute  $z = \frac{1}{x}$ , clear denominators, and find the smallest root. On the other hand, if we know that there is a root near  $z = 2$ , we can substitute  $x = z - 2$  and the process will converge more quickly.

This method of Euler is seldom used today. It is considerably slower than Newton's method, though it does have some advantages over Newton's method, as the reader who is doing calculations by hand may already have discovered. Newton's method requires long division at every step. If the coefficients are simple, as they have been in every example, then Euler's method requires only one long division at the end. The reader is encouraged to work out some more examples and see if there are any other advantages or disadvantages.

References:

- [E101] Euler, Leonhard, *Introductio in analysin infinitorum*, 2 vols., Bosquet, Lucerne, 1748, reprinted in the *Opera Omnia*, Series I volumes 8 and 9. English translation by John Blanton, Springer-Verlag, 1988 and 1990. Facsimile edition by Anastaltique, Brussels, 1967.

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