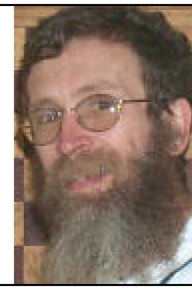
How Euler Did It

by Ed Sandifer

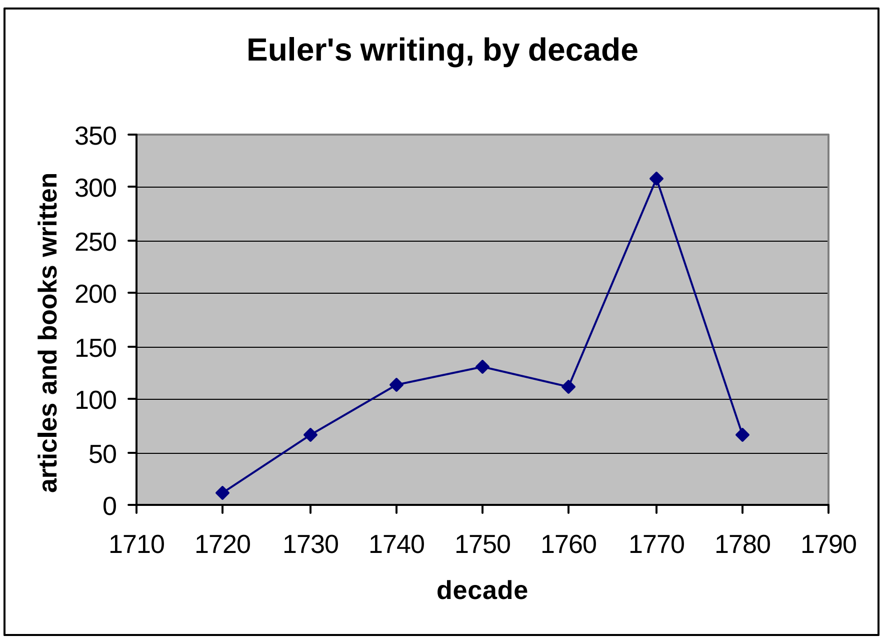


A memorable example of false induction

August 2005

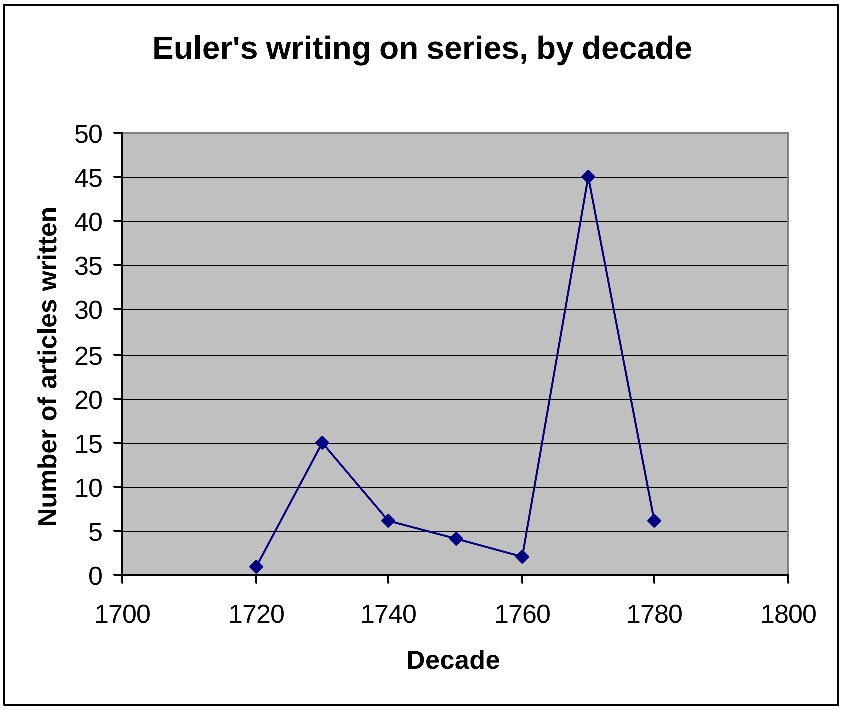
Euler wrote about 800 books and papers. An exact number is hard to define. The “official” number of entries in Eneström’s index is 866, but that includes a number of letters and unfinished manuscripts that Euler never expected to be published. Euler probably intended to finish some of the manuscripts, but others he had probably abandoned as dead-ends. Moreover, though most of Euler’s letters were simple communications, some were more like “open letters,” intended to be shared widely, so they were more like publications than private communication. Taking all of this into account, an estimate of “about 800” publications seems quite reasonable.

Euler wrote his first article in 1725, and it was published in 1726. He died in 1783, but papers intended for publication continued to appear until 1862, 79 years after his death. Below, we give a graph and a table describing the decades that Euler wrote 810 of his books and articles.



Decade	Works
1720	11
1730	67
1740	114
1750	131
1760	111
1770	309
1780	67

Of the 810 books and articles in this database, the Editors of Euler's *Opera Omnia* classified 81 of them, exactly 10%, as being about series, and so published them in volumes 14, 15 and 16 of Series I. The Editors used what some might think is an expanded definition of "series" that also includes infinite products and continued fractions. The timing of Euler's work in series has a somewhat different shape than his work as a whole, as seen in the graphics below.



Decade	Series papers
1720	1
1730	15
1740	6
1750	4
1760	2
1770	45
1780	6

Euler's interest in series seems to be declining through the heart of his career, in the 1740's and 1750's, to the point where he wrote only two papers on the subject in the whole of the 1760's. One of those papers was on properties of the Bernoulli numbers, and the other, the one we discuss here, on properties of a particular series.

This column's paper is E-326, written in 1763 and titled *Observationes analyticae*, or "Analytical observations." Euler plans to sum the middle terms of powers of quadratics, starting with the very simple quadratic, $1 + x + xx$. He begins by listing the powers of $1 + x + xx$:

$$\begin{gathered}\mathbf{1}\\ 1 + \boldsymbol{x} + xx\\ 1 + 2x + \mathbf{3x^2} + 2x^3 + x^4\\ 1 + 3x + 6x^2 + \mathbf{7x^3} + 6x^4 + 3x^5 + x^6\\ \text{etc.}\end{gathered}$$

Now we look at the terms

$$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ \text{etc.}$$

The *Encyclopedia of Integer Sequences*, [EIC] calls the coefficients "central trinomial coefficients." Euler wants to know the rules that give these numbers.

He begins by rewriting

$$(1+x+xx)^n = \left(x(1+x)+1\right)^n.$$

He expands the right hand side as a binomial, getting

$$x^n(1+x)^n + \frac{n}{1}x^{n-1}(1+x)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2} + \text{etc.}$$

This is just the Binomial Theorem. It would look more familiar if Euler had written this paper just a few years later, after he introduced an almost modern notation for binomial coefficients, writing $\left(\frac{n}{k}\right)$ where we would usually write $\binom{n}{k}$.

This last expression still contains binomials, so Euler steadfastly expands it again and combines like terms to find that the coefficient of x^n is

$$1+\frac{n(n-1)}{1\cdot 1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}+\text{etc.}$$

Armed with this formula, Euler calculates the first 12 terms of his sequence. If he had access to the online *Encyclopedia of Integer Sequences*, then in just a few moments he could have found over 20 terms:

1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953, 25653, 73789, 212941, 616227, 1787607, 5196627, 15134931, 44152809, 128996853, 377379369, 1105350729, 3241135527, 9513228123, 27948336381, 82176836301, 241813226151, …

Having found a direct formula for the central trinomial coefficients, and listing the first twelve coefficients, up to 73789, Euler begins a section mysteriously titled:

EXEMPLUM MEMORABILE INDUCTIONIS FALLACIS

Formulas are sometimes cumbersome, and this formula is particularly so. True to form, Euler sets out to find a recursive formula for these numbers. He writes his sequence in one row, the triple of the sequence, offset by one position, in the second row, and subtracts the first row from the second. It looks like this:

1	1	3	7	19	51	141	393	1107	3139	etc.
	3	3	9	21	57	153	423	1179	3321	etc.
	2	0	2	2	6	12	30	72	182	etc.

Now Euler notices (*non sine ratione evenire videtur*, "not without thought it is seen to turn out that) all the numbers in this last row are the double of triangular numbers, and so have the form $mm + m$, for various values of m. Some people used to call these products of consecutive integers of the form $m(m + 1)$ *oblong* or *Pronic* numbers, but Euler does not use these terms.

So, what values of m give these particular values if $mm + m$? Euler calls these values of m the *indices*, and the indices go

1, 0, 1, 1, 2, 3, 5, 8, 13, etc.

This is the Fibonacci sequence, starting just a little bit early, with first two terms 1 and 0, rather than the more familiar starting point 1 and 1. Actually, this apparently wasn't *called* the Fibonacci sequence until the late 1800's, but that wouldn't keep Euler from knowing a lot about the sequence. In particular, he knows from his work on difference equations and generating functions that the nth term of this sequence is given by the formula

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

Let us do something Euler couldn't do, because subscripted sequences hadn't been invented yet, and denote these values by f_n.

From this, Euler can deduce a recursive formula. If we write the sums of the central trinomial terms as

$$1+x+3x^2+7x^3+19x^4+\ldots+Px^n+Qx^{n+1}+\text{etc.}$$

then for the data in this table $3P - Q = (f_n + 1)f_n$.

Euler also derives a direct formula for P, and a second recursive formula that is homogeneous (i. e. one that does not involve n.)

With these, we can find central trinomial coefficients quickly and easily, and we get

1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8955, 25675, 73945, etc.

But wait a minute! This isn't the same sequence we started with! It is the same for the first nine terms, up to 3139, but for the tenth term, this has 8955 where there should be an 8953, and after that the differences become even larger. Now we see the meaning of the title of this section, which translates as "A notable example of false induction." He had warned us. There really are two different sequences, each defined by reasonable and interesting patterns that agree for the first nine terms, and then become different.

Euler still has an article to finish, but nothing else this interesting happens. He finds the correct recursive relation directly from the formulas (so it is correct): if P, Q and R are consecutive coefficients, then the nth term is given by the relation

$$R=Q+\frac{n+1}{n+2}(Q+3P).$$

That done, he spends the rest of the article by investigating the central coefficients of powers of general quadratics of the form $a + bx + cxx$.

In 1753, Euler had written an E-256, *Specimen de usu observationum in mathesi pura*, "Example of the use of observation in pure mathematics." It was an article about number theory, showing how experiments on integers of the form a^2+2b^2 led him to observe that such forms are closed under multiplication. This told him what to try to prove, and soon led to a proof of that and several related results.

Ten years later, he gives us a graphic illustration of the limits of observation, and that it shows mathematicians what might be true, not necessarily what is true.

References:

[E256] Euler, Leonhard, Specimen de usu observationum in mathesi pura, Novi commentarii academiae scientiarum Petropolitanae, **6** (1756/7) 1761, pp. 185-230, reprinted in *Opera Omnia* Series I vol 2 pp. 459-493, available on line at EulerArchive.org.

[E326] Euler, Leonhard, Observationes analyticae, Novi commentarii academiae scientiarum Petropolitanae, **11** (1765) 1767, pp. 124-143, reprinted in *Opera Omnia* Series I vol 15 pp. 50-69.

[EIS] Encyclopedia of Integer Sequences, on line at http://www.research.att.com/~njas/sequences/, consulted July 21, 2005.

[W] Weisstein, Eric W. "Pronic Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/PronicNumber.html.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 33 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.