How Euler Did It

by Ed Sandifer

Orthogonal matrices

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Jeff Miller's excellent site [M] "Earliest Known Uses of Some of the Words of Mathematics" reports:

> "The term **MATRIX** was coined in 1850 by James Joseph Sylvester (1814-1897):
>
>> [...] For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of *m* lines and *n* columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number *p,* and selecting at will *p* lines and *p* columns, the squares corresponding of *p*th order.
>
> The citation above is from "Additions to the Articles *On a new class of theorems,* and *On Pascal's theorem,*" Philosophical Magazine, pp. 363-370, 1850. Reprinted in Sylvester's *Collected Mathematical Papers,* vol. 1, pp. 145-151, Cambridge (At the University Press), 1904, page 150."

On the subject of orthogonal matrices, he writes

> The term **ORTHOGONAL MATRIX** was used in 1854 by Charles Hermite (1822-1901) in the *Cambridge and Dublin Mathematical Journal,* although it was not until 1878 that the formal definition of an orthogonal matrix was published by Frobenius (Kline, page 809).

Imagine my surprise when I was browsing Series I Volume 6 of Euler's *Opera Omnia* trying to answer a question for Rob Bradley, President of The Euler Society. This is the volume *Ad theoriam aequationum pertinentes*," "pertaining to the theory of equations." In the middle of that volume I found an article [E407] with the unremarkable title "Algebraic problems that are

memorable because of their special properties," *Problema algebraicum ob affectiones prorsus singulares memorabile*. Usually, it seems, when Euler calls a problem "memorable," I don't agree. This was an exception.

Euler asks us to find nine numbers,

$$A, \quad B, \quad C,$$
$$D, \quad E, \quad F,$$
$$G, \quad H, \quad I$$

that satisfy twelve conditions:

1.	$A^2 + D^2 + G^2 = 1$		4.	$AB + DE + GH = 0$
2.	$B^2 + E^2 + H^2 = 1$		5.	$AC + DF + GI = 0$
3.	$C^2 + F^2 + I^2 = 1$		6.	$BC + EF + HI = 0$
7.	$A^2 + B^2 + C^2 = 1$		10.	$AD + BE + CF = 0$
8.	$D^2 + E^2 + F^2 = 1$		11.	$AG + BH + CI = 0$
9.	$G^2 + H^2 + I^2 = 1$		12.	$DG + EH + FI = 0$

If we regard the nine numbers as a 3x3 matrix

$$M = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

then conditions 1, 2, 3, 10, 11 and 12 are exactly the conditions that make $MM^T = I$. In modern terms, this makes M an orthogonal matrix.

Note that the other six conditions make $M^T M = I$, another characterization of orthogonal matrices.

Of course, Euler doesn't *know* these are orthogonal matrices. When he wrote this paper in 1770, people didn't use matrices to do their linear algebra. As Jeff Miller's site suggests, Euler was doing this 80 years before these objects joined the mathematical consciousness.

What, then, was Euler thinking when he formulated this problem? We can only speculate. It was probably something more than that he admired the pretty patterns, but something less than that he sensed the profound power of operations on such arrays of numbers.

Let's look at Euler's paper. First he notes that we have 12 equations, but only 9 unknowns, so there is a chance that the problem has no solution. Rather than showing that the system of equations has a solution by giving an example of a solution (say $A = E = I = 1$, all the rest of the unknowns equaling zero), he offers a theorem:

Theorem: If nine numbers satisfy the first six conditions given above, then they also satisfy the other six.

Euler himself describes the proof of this theorem as *calculos vehementer intricatos*, "vehemently intricate calculations," and we won't describe them in any significant detail. He does do an interesting step at the beginning, though.

Euler re-writes conditions 4, 5 and 6 as

4. $AB = -DE - GH$
5. $AC = -DF - GI$
6. $BC = -EF - HI$

Then he asks us to multiply equation 4 by equation 5, then divide by equation 6 to get, as he writes it,

$$\frac{4 \cdot 5}{6} : \frac{AABC}{BC} = AA = -\frac{(DE + GH)(DF + GI)}{EF + HI}.$$

The notation $\dfrac{4 \cdot 5}{6}$ is not an arithmetic operation, but an *ad hoc* notation for an algebraic operation on equations 4, 5 and 6. Euler does this one or two times in other papers, but this is the only time he uses such a notation in this paper.

After three pages of such calculations, Euler eventually derives all of the conditions 7 to 12 from conditions 1 to 6, thus proving his theorem.

Now Euler turns to the "solution of the problem that was proposed at the beginning." Condition 1 (or 7) guarantees that A is between –1 and 1, so it has to be the cosine of something. Let $A = \cos z$. (Note that Euler still uses cos. as an abbreviation for "cosine," hence the period. Now that we've mentioned it, we'll write it the modern way. We'll also write $\sin^2 z$ where Euler wrote $\sin z^2$.)

Then, from conditions 1 and 7, we get

$$DD + GG = 1 - AA = 1 - \cos^2 z = \sin^2 z$$

and similarly

$$BB + CC = \sin^2 z.$$

These equations will be satisfied by taking

$$B = \sin z \cos h \qquad C = \sin z \sin h \qquad D = \sin z \cos q \qquad G = \sin z \sin q.$$

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Let's check the score. We have six equations, nine unknowns, and we've made three arbitrary decisions by choosing *z,*, *h* and *q*. Euler knows enough linear algebra (see, for example, [S 2004]) to suspect that this will determine a unique solution. Indeed, they determine *A, B, C, D* and *G*. Indeed, with half a page of calculations he first finds that

$$E = \sin h \sin q - \cos z \sin h \cos q \text{ and}$$
$$I = \cos h \cos q - \cos z \sinh h \sin q$$

and then that

$$F = -\cos h \sin q - \cos z \sin h \cos q \text{ and}$$
$$H = -\sin h \cos q - \cos z \cos h \sin q .$$

This solves the problem. In typical Eulerian fashion, though, he doesn't stop there. Instead, he develops some slightly more efficient techniques and goes on to solve analogous problems for 4x4 and 5x5.

Almost as if Euler did not want us to believe that he was actually doing modern linear algebra, Euler's last problem is to find a 4x4 array of integers

$$
\begin{array}{cccc}
A & B & C & D \\
E & F & G & H \\
I & K & L & M \\
N & O & P & Q
\end{array}
$$

(he skipped "J" on purpose) satisfying the 12 orthogonality equations

$$AE + BF + CG + DH = 0$$
$$AI + BK + CL + DM = 0$$
$$\text{etc.}$$

and the additional condition on the sums of the squares of the numbers on the two diagonals,

$$A^2 + F^2 + L^2 + Q^2 = D^2 + G^2 + K^2 + N^2 .$$

He demonstrates how to find two different solutions. One is

$$
\begin{array}{cccc}
68 & -29 & 41 & -37 \\
-17 & 31 & 79 & 32 \\
59 & 28 & -23 & 61 \\
-11 & -77 & 8 & 49
\end{array}
$$

where the dot product of any row (column) with any other row (column) is zero, and the sum of the squares of 68, 31, -23 and 49 equals the sum of the squares of –37, 79, 28, -11. They both equal 8415.

His other solution is

$$
\begin{array}{rrrr}
73 & -85 & 65 & -11 \\
-53 & 31 & 107 & 41 \\
-89 & -67 & 1 & -67 \\
-29 & -65 & -35 & 103
\end{array}
$$

where the sums of the squares on the diagonals are 16,900. Euler's methods are typical of his work in Diophantine equations, and would allow us to generate as many solutions as we want.

References:

[E407]   Euler, Leonhard, Problema algebraicum ob affections prorsus singulars memorabile, *Novi Commentarii academiae scientiarum Petropolitanae* **15**, (1770) 1771, p. 75-101 , reprinted in *Opera Omnia* Series 1, Volume 6, pp. 287-315. Available through The Euler Archive at www.EulerArchive.org.
[M]   Miller, Jeff, Earliest Uses of Symbols in Number Theory, http://members.aol.com/jeff570/nth.html, consulted June 13, 2006.
[S 2004] Sandifer, Ed, Cramer's Paradox, *How Euler Did It*, MAA Online, August 2004.

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How Euler Did It is updated each month.