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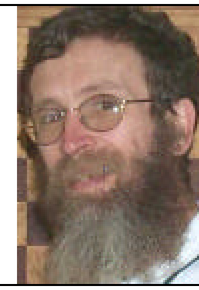
How Euler Did It

by Ed Sandifer



Piecewise functions

January 2007

This year, 2007, marks the 300th anniversary of Euler’s birth on April 15, 1707. We begin our celebration of Euler’s birthday by discussing one of Euler’s most fundamental contributions to mathematics, the idea of a function.

The word “function” comes to us from the Latin *functio*, meaning a performance, an event or an activity, not, as we might hope, from the German *der Funke*, a spark or a glimmer. (The colloquial “funky” comes from the German.)

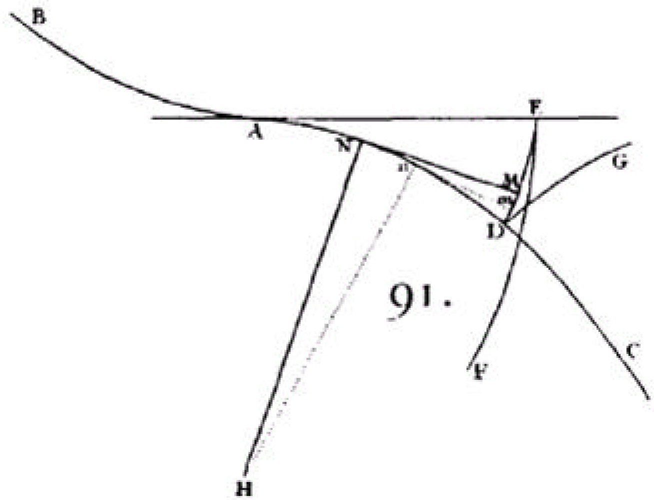
Today, functions are one of the central objects in mathematics. David Hilbert told us, “Besides the concept of number, the concept of function is the most important one in mathematics.” [T] On the other hand, Hilbert’s student, Hermann Weyl wrote, “Nobody can explain the function concept:” [T]

The Ancients knew some of the relations between curves and algebraic expressions. Both Apollonius and Archimedes, for example, knew how the shape of a parabola was related to the algebraic expression $ay = x^2$, though they didn’t use algebraic notation to express the relation. They were philosophically and notationally unable to make sense of an expression like $y = x^2$ because one object in the expression, *y*, is a length and the other, x^2 represents an area. They regarded relations like $ay = x^2$ as properties of curves, and not as definitions of the curves themselves, and they called such properties *symptoms.*

In the early 1600’s, Descartes devoted a big piece of his *Geometria* to giving meaning to nonhomogeneous expressions like $y = x^2$. Soon the idea developed that every curve had an associated algebraic expression of some sort, but the formula was still regarded as a property of the curve. Formulas were not yet stand-alone objects.

Functions gradually earned their own identity as the 18th century progressed. In 1696, when L’Hôpital wrote *Analyse des infiniment petits pour l'intelligence des lignes courbes*, the world’s first calculus book, he wrote about curves, and a curve existed if it could be constructed by some mechanical or geometric process. Fifty years later, Euler wrote the *Introductio in analysin infinitorum* from the point of view of functions, and a curve existed if it could be described by an analytic expression. In fact, Rob Bradley [B] describes an interesting story contrasting the two ideas of what makes a curve.

L'Hôpital had described a "cusp of the second kind" sometimes called a "bird's beak." L'Hôpital was studying involutes of curves like the curve *BANDC* shown at the right in his Figure 91. The curve has an inflection point at *A*. The involute of the curve is shown as the awkwardly named curve *DMFF*, which, at its point *F* corresponding to the inflection point *A*, has a cusp for which both branches curve the same direction, like a bird's beak. Hence the name.



In 1696, people had no problem accepting that such curves existed. There was a clear mechanical construction. By 1740, though, people weren't so sure, since they couldn't seem to find an analytical representation of such curves. In 1748 in the *Introductio,* Euler gave a formula, and the bird's beak was restored. It was curious that people believed their formulas more than they believed their eyes.

We may return to this episode in some future column.

Euler was a bit like Hermann Weyl when it came to the function concept itself. Euler knew what he wanted functions to *do*, but he sometimes struggled to articulate what they *are*. Early on, a function was an analytic expression describing a curve. In an expression like $x^2 + y^2 = 1$, x is a function of y, but y is also a function of x, since knowing one, x or y, we can determine the other. Euler also allowed multi-valued functions. For example, in the expression $y = x^2$, y is a single-valued function of x, but x is a multi-valued function of y.

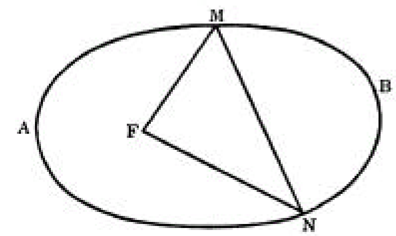
Euler also accepted solutions to differential equations as functions, even if those solutions might not be written down explicitly. Of course, he was quite unaware of the exotic "pathological" functions that Weierstrass and Dirichlet would describe in the 19th century.

Euler was not always consistent as he struggled to refine the concept of a function. As an example of this struggle, we will take a closer look at Euler's thoughts about what we now call "piecewise defined functions."

Euler usually thought that a function had to be defined by the same analytical expression everywhere. Since he did not have any notation for the absolute value function, perhaps the best-known piecewise-defined function, he never had cause to realize that a function as natural as the absolute value function is actually defined piecewise. He occasionally came across the absolute value function disguised as $\sqrt{x^2}$, but when he did, he was always interested in other issues.

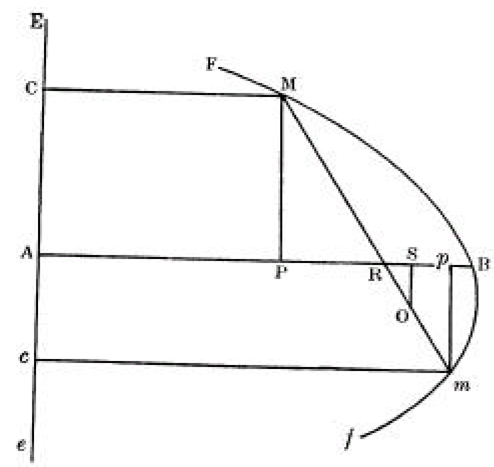
He began rejecting piecewise functions early in his career. Two of his earliest papers, E3, "Methodus inveniendi traiectorias reciprocas algebraicas" and E5, "Problematis traiectoriarum reciprocarum solution," deal with the now forgotten and misunderstood topic of "reciprocal trajectories," curves with a peculiar kind of symmetry that people sometimes incorrectly believe has something to do with ballistic trajectories. Reciprocal trajectories are somewhat esoteric, and rather than investing the time to explain them, we'll jump forward a few years to the fruits of one of Euler's shortest papers.

In 1745, Euler sent a short note to *Nova Acta eruditorum* forwarding a problem posed anonymously by Christian Goldbach. The note became an eight-line "paper" [E79] titled "A problem of geometry proposed publicly by an anonymous geometer," probably Euler's shortest paper and maybe one of the shortest mathematics paper anyone ever wrote. In E79, Euler and Goldbach, referring to the figure at the right, ask what curves like *AMBN* there might be with the property that there is a point *F* from which any ray, like *FM*, reflected twice, returns to the point *F*.

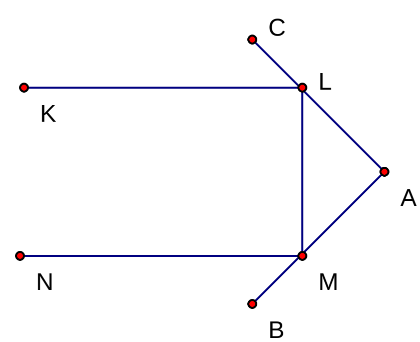


Ellipses have this property. The point *F* can be either of the foci of the ellipse. It is a familiar property of ellipses that any ray from one focus reflects to pass through the other focus. There it becomes a ray from a focus, so it will reflect again and return to the first focus. Goldbach and Euler ask if there are any other such curves, or if this is in itself a defining property of an ellipse?

In E79, Euler only posed the problem, but he solved it two years later in E106, "Solution to the catoptric problem in *Novis Actis Eruditorum Lipsiensibus* proposed in November 1745." He found that there were, indeed, curves other than the ellipse with this special property, and then, in typical Eulerian style, he turned to variations of the same problem. He looked at a problem that is projectively related to the ellipse-like problem he started with. He sought to find if the parabola is the only curve like *FMBmf*, shown in the illustration at the right, with the property that rays like *CM*, parallel to the axis of the curve *AB* reflected twice, as *Mm*, then *mc*, will give a ray *mc* parallel to the original.



Much like ellipses, parabolas have this property. The reflection of the ray *CM* will pass through the focus of the parabola, shown in the figure as the point *R* on the axis *AB* of the parabola. Euler asks if there are any other such curves, and, of course he finds some. In the course of his analysis, though, he explicitly rejects curves like the one shown in the next figure, where the two branches are not described by the same analytic expression. In this figure, the branch *AC* is perpendicular to the branch *AB*, and the rays *KL* and *MN* are parallel to the bisector of angle *CAB*.



There was a philosophical basis for rejecting curves like the absolute value function. Leibniz championed something usually called the "principle of continuity," but since the word "continuity" means something different to mathematicians, we'll call it the "principle of continuation."[1] Broadly speaking, the principle of continuation says that similar things will behave similarly. Leibniz summarized the principle writing "Nature makes no leaps." We have seen Euler use the principle of continuation before when he does arithmetic with infinite and infinitesimal "numbers" just like they were ordinary finite numbers.

[1] Of course, the name "principle of continuation" has its own meaning in other contexts. For example, to geologists, it describes a property of layers of sediments.

In the present problems, the principle of continuation tells us that the solution to a "natural" problem will not have any "leaps" in it, and it will be described by a single analytic expression.

Given this world view, it would be surprising if Euler proposed a piecewise function as the solution to a "natural" problem. Yet he did exactly that when he analyzed ballistic trajectories. In the article we described in last month's column [E217] he tells us that the forces acting on a cannonball (*not* neglecting air resistance) give different differential equations for the ascending branch than for the descending branch. In particular, Euler takes x and y coordinates as usual, s to be arc length, t to be time, and c is a parameter describing the properties of air. The variable v, though, might be confusing to the modern reader. It is the height from which an object would have to be dropped to have the same speed as the cannon ball has at time t. Hence, v is a length, not a velocity or a speed, and because of his choice of units, the speed, $\frac{ds}{dt}$ is given by $\frac{ds}{dt} = \sqrt{v}$. Since air resistance is taken to be proportional to the square of the speed, this makes air resistance proportional to v itself. Euler also takes a to be the acceleration due to gravity.

With this notation in place, Euler resolves forces and finds that the acceleration in the x direction is given by the same differential equation,

$$\frac{2ddx}{td^2} = -\frac{vdx}{cds}$$

whether the cannonball is ascending or descending. In what seems to be a violation of the law of continuation, the acceleration is given by

$$\frac{2ddy}{dt^2} = a - \frac{vdy}{cds}$$

when the cannonball is ascending, but it is given by

$$\frac{2ddy}{dt^2} = a + \frac{vdy}{cds}$$

when it is descending.

I suspect that Euler was not thinking about the law of continuation when he wrote this. If he had, though, he might have tried to explain it by noting that at the apex, where the trajectory changes from its ascending branch to its descending branch, the factor $\frac{dy}{ds}$ gradually vanishes and reappears. The leap isn't in nature, but in our notation.

We can't let Euler off the hook that easily, though. I was careful above to describe the symbol c as a parameter, not a constant. Euler takes e^3 to be the volume of water with the same mass as the cannonball and d to be the diameter of the cannonball. Then he tells us that if the speed of the projectile is not too fast, then

$$c = \frac{2133e^3}{dd}.$$

However, he seems to have done some experiments and concluded that "if the movement is so rapid that air cannot immediately occupy the space behind the globe, then globe will leave behind itself a kind of void space, and so for that instant the globe will be subject to the full pressure of the atmosphere, which will not be counterbalanced by an equal pressure from behind, and so it is clear that the resistance will be augmented by the entire pressure of the atmosphere on the part at the front of the globe." Euler calculates that this changes the air resistance from $\frac{v}{c}$ to $\frac{v}{c}+\frac{6666k}{4c}$, where k is the air pressure measured in feet of water.

This is a sudden change in force that Euler's data indicates occurs when $v > 28050$ feet. Euler tells us this translates to 1325 feet per second. Modern theory puts this change close to the speed of sound, 1087 feet per second. Euler knew the speed of sound fairly accurately, so he apparently didn't understand how the speed of sound is related to this phenomenon.

Euler apparently did not try to reconcile this sudden change in force with the principle of continuation. After all, the theory worked to describe trajectories. The void space behind the cannonball seemed to explain the phenomenon, even if the analytic representation makes that troublesome leap.

I think that, in not committing too strongly to the principle of continuation, Euler displayed an admirable lack of rigor that left the concept of a function with enough flexibility that it could evolve into the foundation of mathematics it has become today.

References:

[B] Bradley, Robert E., The Curious Case of the Bird's Beak, *International Journal of Mathematics and Computer Science*, vol. 1, no. 2, (2006), p. 243-268.

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[T] Thiele, Rüdiger, The Function Concept, from an address accepting the Lester R. Ford award, MathFest, Providence, RI, August 2004.

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How Euler Did It is updated each month.