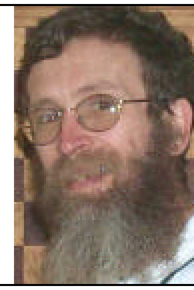
How Euler Did It

by Ed Sandifer



Gamma the function

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Euler gave us two mathematical objects now known as "gamma." One is a function and the other is a constant. The function, $\Gamma(x)$, generalizes the sequence of factorial numbers, and is the subject of this month's column. A nice history of the gamma function is found in a 1959 article by Philip Davis, [D] and a shorter one is online at [Anon.]. The second gamma, denoted γ, is a constant, approximately equal to 0.577, and, if things go as planned, it will be the subject of next month's column. In 2003, Julian Havel wrote a book about gamma the constant. [H]

When Euler arrived in St. Petersburg in 1728, Daniel Bernoulli and Christian Goldbach were already working on problems in the "interpolation of sequences." Their problem was to find a formula that "naturally expressed" a sequence of numbers. For example, the formula n^2 "naturally expresses" the sequence of square numbers, 1, 4, 9, 16, …, and $\frac{n(n+1)}{2}$ expresses the sequence of triangular numbers, 1, 3, 6, 10, 15, … . Both of these are well defined for fractional values of *n*, so they were said to *interpolate* the sequences.

Earlier mathematicians including Thomas Harriot and Isaac Newton had developed an extensive calculus of finite differences to help find formulas that matched various sequences of values, and their work helped lead to the invention of calculus. In fact, one way to understand the discovery of logarithms is that they resulted from the interpolation of geometric series.

Bernoulli and Goldbach were stumped trying to interpolate two particular sequences. The first was the sequence we now call the factorial numbers, 1, 2, 6, 24, 120, 720, etc. They called it the "hypergeometric progression." The second was the sequence of partial sums of the harmonic series, 1, $1+\frac{1}{2}$, $1+\frac{1}{2}+\frac{1}{3}$, etc.

Shortly after he learned of the problems, Euler solved both of them. This month we are interested in his solution of the first one, in which he showed us how to give meaning to expressions like $\left(2\frac{1}{2}\right)!$, as a natural interpolation between 2! = 2 and 3! = 6.

Euler announced his solution in a letter to Christian Goldbach dated October 13, 1729. He began his letter, "Most Celebrated Sir: I have been thinking about the laws by which a series may be interpolated. … The most Celebrated [Daniel] Bernoulli suggested that I write to you." He goes on to proclaim that the general term of the "series" 1, 2, 6, 24, 120, etc. (at the time, people used the words series, sequence and progression interchangeably) is given by

$$(1)\qquad \frac{1\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\text{ etc.}$$

There is a subtle reason that Euler used the form given in (1) rather than the more "obvious" form

$$(2)\qquad \frac{1}{1+n}\cdot\frac{2}{2+n}\cdot\frac{3}{3+n}\cdot\frac{4}{4+n}\text{ etc.}$$

The reason involves absolute convergence of infinite products, and something called "Gauss's criterion." In fact, (1) converges as it is written, as the limit of its partial products, but to make (2) converge, we must interpret it as

$$(3)\qquad \lim_{k\to\infty}\frac{1}{1+n}\cdot\frac{2}{2+n}\cdot\frac{3}{3+n}\cdot\cdots\cdot\frac{k}{k+n}\cdot k^n,$$

or else the limit will be zero. We won't go into details here, but instead refer the reader to Walker [W]. Euler clearly knew that something like Gauss's criterion was necessary when he made his definition, but then he doesn't use the criterion much in his exposition.

Euler's exposition in the letter of October 1729 is very brief, but he gave more details and consequences in an article, De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt, "On transcendental progressions, or those for which the general term is not given algebraically." [E19] In the article he tells us, without doing the calculations, that if n is 0 or 1, then the product is 1. For n equal to 2 and 3, he gives us a little more, telling us that

$$\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\text{etc.}, \text{ which he says equals 2,}$$

and

$$\frac{2\cdot 2\cdot 2}{1\cdot 1\cdot 4}\cdot\frac{3\cdot 3\cdot 3}{2\cdot 2\cdot 5}\cdot\frac{4\cdot 4\cdot 4}{3\cdot 3\cdot 6}\cdot\frac{5\cdot 5\cdot 5}{4\cdot 4\cdot 7}\cdot\text{etc.}, \text{ which equals 6.}$$

As a more complicated example, Euler takes $m = 1/2$ and gets an infinite product:

$$\frac{1}{1+1/2}\cdot\frac{2}{2+1/2}\cdot\frac{3}{3+1/2}\cdot\frac{4}{4+1/2}\cdot\text{etc.}$$

This, in turn, equals

$$(4)\qquad \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\text{etc.}$$

It seems hopeless to try to evaluate (4), but even at age 22, Euler has read a great deal of the mathematical literature. In particular, he knew that in 1665, John Wallis had found that

$$\frac{p}{4} = \frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot \text{etc.}$$

From this, it is not hard to find that the value given in (4) is $\frac{\sqrt{p}}{2}$. Philip Davis [D] speculates that Euler recognized a connection between p and areas and integration. Then, when he saw that the value of his infinite product involved p, he thought to try to rewrite the infinite product as an integral. After a good deal of work, described in detail both in [D] and in [S], he finds that his infinite product equals

$$\int_0^1 (\ln x)^n \, dx$$

and that this, too, is well defined for fractional values of n as well as for negative values of n that are not integers.

Near the end of the paper, Euler proposes an "application", though he admits that the example might not be very useful. He writes:

> "To round off this discussion, let me add something which certainly is more curious than useful. It is known that $d^n x$ denotes the differential of x of order n and if p denotes any function of x and dx is taken to be constant then … the ratio of $d^n p$ to dx^n can be expressed algebraically. … We now ask, if n is a fractional number, what the value of that ratio should be."

Euler is proposing that we use his new function to find what we now call "fractional derivatives", and he gives us some examples. For this, we will use modern notation, and use what we now call the gamma function, denoted $\Gamma(x)$. We note that if x is a non-negative integer, then $\Gamma(x+1) = x!$. Let's have a look at some of the elementary properties of k-th derivatives of x^n and watch for a pattern:

first derivative	nx^{n-1},
second derivative	$n(n-1)x^{n-2}$,
third derivative	$n(n-1)(n-2)x^{n-3}$,
…	
k-th derivative	$n(n-1)(n-2)\cdots(n-k+1)x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$,

where we have to be a little careful that $k \leq n$ so that $n - k \geq 0$ and so $(n - k)!$ will be defined. This problem goes away if k is not an integer.

Armed with the gamma function, now we can define the k-th derivative even if k is a fraction, as follows:

k-th derivative of x^n is $\frac{\Gamma(n+1)}{\Gamma(n-k+1)} x^{n-k}$,

or, more like the way Euler wrote it,

$$\frac{d^k(z^e)}{dz^n} = z^{e-n} \frac{\int_0^1 (-\ln x)^e \, dx}{\int_0^1 (-\ln x)^{e-n} \, dx},$$

where, to Euler, e is just an exponent, and does not yet have today's connotations as a special constant.

Using this idea, Euler takes $n = 1$ and $k = ½$, to find that the ½-th derivative of x is

$$\frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{1}{\sqrt{p}/2}\sqrt{x}$$
$$= \frac{2\sqrt{x}}{\sqrt{p}}$$

Euler does not take this any farther, but it is easy for us to see at least one way to do it. If we would like to take fractional derivatives of some more complicated function $f(x)$, then we can try to take the Taylor series for f, and apply Euler's fractional derivative formula to each term. Readers with the skills and the software are encouraged to experiment with this with their favorite mathematical software like Maple ™ or Mathematica. ™ One good place to start might be to take a polynomial like $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$ and build an animation of the graph of its k-th derivatives, say for x between 0 and 6, as k increases from 0 to 5. We know that f itself has 5 roots, its first derivative has 4, its second has 3, etc. It is interesting to watch what happens to the roots as k increases, until, after five derivatives, all the roots disappear.

Trigonometric functions, like $f(x) = \sin x$ are also interesting.

Readers who know about the properties of Fourier series and Laplace transforms may know that they, too, can be used to define fractional derivatives.

References:

[Anon] Gamma Function, http://en.citizendium.org/wiki/Gamma_function, August 23, 2007.

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[W] Walker, Peter J., *Elliptic Functions: A Constructive Approach*, Wiley, New York, 1996.

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How Euler Did It is updated each month.