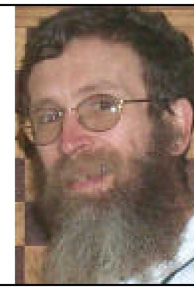
How Euler Did It

by Ed Sandifer



Multi-zeta functions

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Two of Euler's best known and most influential discoveries involve what we now call the Riemann zeta function. The first of these discoveries made him famous when he solved the Basel problem. He showed [E41] that the sum of the reciprocals of the square numbers was

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\text{etc.}=\frac{\pi^2}{6}.$$

Euler's second great result [E72] on this topic was what we now call the Euler product formula, and we write it as

$$\sum_{k=1}^{\infty}\frac{1}{k^n}=\prod_{p\text{ prime}}\frac{1}{1-1/p^n}.$$

For the readers unfamiliar with the zeta function, we'll give a brief introduction.

It has long been known that the harmonic series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, diverges, but that if we take these terms to some power, as $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots$, then the series will converge whenever $n > 1$. The value to which it converges depends on n, and is now denoted $\zeta(n)$.

Euler's first result showed that $\zeta(2)=\frac{\pi^2}{6}$, and his second result showed that $\zeta(n)$ can be written either as an infinite sum or as an infinite product.

Since Euler's time, the zeta function has captured the imaginations of many great mathematicians. In particular, in 1859 Bernhard Riemann, showed that n need not be a real number, and that the zeta function has a natural analytic continuation as a function of a complex variable. Hence the function is traditionally called the *Riemann* zeta function and defined in terms of a complex variable s as

$$z\left(s\right)=\sum_{k=1}^{\infty}\frac{1}{k^s}.$$

Riemann surmised that his function is zero for infinitely many values of s, and that all its complex roots share the property that their real part is ½. For some reason that is still unclear, Riemann's conjecture is known as the "Riemann hypothesis" instead of the "Riemann conjecture." Though it is badly named, it is one of the most important unsolved problems in mathematics today.

Over the years, zeta functions have evolved a number of variations. For example, instead of taking the sum over the ordinary integers, one could take the sum over the integers in some number field. This leads to a topic known as L-series. We could also change the numerators in the sum, and look at sums like

$$\sum_{k=1}^{\infty}\frac{c\left(k\right)}{k^n},$$

where $c\left(k\right)$ is some function of k. We saw Euler himself do something like this in last month's column, where we describe a series Euler investigated the end of [E190],

$$s=\frac{1}{a}+\frac{2}{a^2}+\frac{2}{a^3}+\frac{3}{a^4}+\frac{2}{a^5}+\frac{4}{a^6}+\frac{2}{a^7}+\frac{4}{a^8}+\frac{3}{a^9}+\text{etc.}$$

This is not exactly an L-series, because the denominators form a geometric series, not an arithmetic series, but the nth numerator is given by the number-theoretic function

$$c\left(n\right)=\text{the number of divisors of } n.$$

This is very much in the spirit of a modern L-series.

At a recent section meeting of the MAA, Michael Hoffmann of the US Naval Academy in Annapolis brought to my attention another modern variation of the zeta function, and showed how that variation derived from Euler's work. Most of the remainder of this column is based on what he showed me. [H1992, H2007]

In the modern way, a *multiple zeta value* is defined as

$$z\left(i_1,i_2,\cdots i_k\right)=\sum_{n_1>n_2\cdots n_k\geq 1}\frac{1}{n_1^{i_1}\cdot n_2^{i_2}\cdots n_k^{i_k}}.$$

Both the motivation and the notation are obscure here. Let's try to untangle both of them at the same time. Let's ask, what would happen if we multiplied together two ordinary zeta functions, say $z\left(m\right)$ and $z\left(n\right)$? As series, we would get

$$z\left(m\right)\cdot z\left(n\right)=\left(\sum_{k=1}^{\infty}\frac{1}{k^m}\right)\cdot\left(\sum_{k=1}^{\infty}\frac{1}{k^n}\right)$$

Euler would not have used the Sigma notation, so he might have written this as

$$\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\text{etc.}\right)\cdot\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}\right)$$

Then he probably would have expanded this to get something like

$$\begin{array}{l} 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\cdots \\ \frac{1}{2^m}+\frac{1}{2^m}\frac{1}{2^n}+\frac{1}{2^m}\frac{1}{3^n}+\frac{1}{2^m}\frac{1}{4^n}+\frac{1}{2^m}\frac{1}{5^n}+\cdots \\ \frac{1}{3^m}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{3^m}\frac{1}{3^n}+\frac{1}{3^m}\frac{1}{4^n}+\frac{1}{3^m}\frac{1}{5^n}+\cdots \\ \frac{1}{4^m}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{4^m}\frac{1}{4^n}+\frac{1}{4^m}\frac{1}{5^n}+\cdots \\ \quad\cdots \end{array} \tag{1}$$

Now we can take this apart and put it back together a different way. First, note the terms on the "diagonal" of this sum, the first term in the first row, the second in the second row, etc. They sum to form an ordinary zeta function as follows:

$$1+\frac{1}{2^m}\frac{1}{2^n}+\frac{1}{3^m}\frac{1}{3^n}+\frac{1}{4^m}\frac{1}{4^n}+\frac{1}{5^m}\frac{1}{5^n}+\cdots=z\,(m+n).$$

On the other hand, the terms below the diagonal sum as

$$\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{4^m}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{5^m}+\text{etc.}$$

This might be a little clearer (or maybe not) if we explicitly include the factors of 1^n in the products. This gives

$$\frac{1}{2^m}\frac{1}{1^n}+\frac{1}{3^m}\frac{1}{1^n}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{1^n}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{5^m}\frac{1}{1^n}+\text{etc.}$$

Now we can see clearly that in each denominator, $a^m b^n$ we have $a > b \geq 1$, so we can rewrite the sum in modern Sigma notation as

$$\sum_{a>b\geq 1}\frac{1}{a^m b^n}$$

Glancing back up the page, we see that is exactly what Hoffman defines as the multi-zeta value $z\,(m,n)$

Likewise, the terms above the diagonal in our product sum to $z\,(n,m)$. This gives one of the motivations for multi-zeta values. They arise in multiplying values of the zeta function, and lead to the formula

(2) $$\zeta(m)\zeta(n)=\zeta(m+n)+\zeta(m,n)+\zeta(n,m)$$

A slightly different approach involves defining a different multi-zeta value, using $\geq$ instead of >, as

$$\zeta^*(m,n)=\sum_{a\geq b\geq 1}\frac{1}{a^m b^n}.$$

This includes the diagonal terms in the big summation, so $\zeta^*(m,n)=\zeta(m+n)+\zeta(m,n)$ and it leads to a similar formula about products of zeta values:

(3) $$\zeta(m)\zeta(n)=\zeta^*(m,n)+\zeta^*(n,m)-\zeta(m+n).$$

All these are modern ideas and modern notations, and they are well documented in the fine bibliography[1] maintained by Michael Hoffman.

I was surprised to learn that these ideas are not of modern origin, but first came from Christian Goldbach in a letter to Euler dated December 24. 1742. [J+W] There, Goldbach uses 18th century notation to find that

$$\zeta^*(3,1)=\frac{\pi^2}{72} \text{ and } 2\zeta^*(5,1)+\zeta^*(4,2)=\frac{19\pi^6}{5670},$$

though he does not claim to know either $\zeta^*(5,1)$ or $\zeta^*(4,2)$.

Euler responded quickly, though at this time Euler was in Berlin and Goldbach was in Moscow, and the mail was perhaps slower in the middle of the winter. Nonetheless, Euler's letter dated January 19, 1743 contained some additions to Goldbach's results, providing equations for $\zeta^*(3,1)$, $\zeta^*(5,1)$, $\zeta^*(7,1)$ and $\zeta^*(9,1)$ in terms of products of the ordinary zeta function. Not all of Euler's claims are correct, though. Hoffman points out that his claim that

$$\zeta^*(6,2)=2\zeta(3)\zeta(5)-\frac{3}{2}\zeta(4)^2+\frac{1}{4}\zeta(8)$$

is false, and so were a few others. In fact, this one isn't even very close. According to Maple, ™ the left hand side is about 1.6557 and the right hand side is about 0.9868, and Michael Hoffman tells me that nobody has yet found a way to write $\zeta^*(6,2)$ as a polynomial function of ordinary zeta values with rational coefficients, and whether or not one exists is an open and active research question.

Euler and Goldbach exchanged a total of five letters on this subject. In the last one, dated February 26, Euler used properties of these multi-zeta functions to give eighteen-place decimal approximations to $\zeta(n)$ through n = 16.

[1] www.usna.edu/User/math/meh/biblio.html

As usual, Euler could not leave to a letter what he could expand into a paper, but Euler apparently let nearly 30 years pass before he returned to multiple zeta values. His work became [E477], written in 1771 and published in 1776. There he begins by citing his letters with Goldbach, and describing the series that is now denoted by $z*(m,n)$. He wrote

> In commercio litterario, quod olim com Illustrissimo Goldbachio coluerum, inter alias varii argumenti speculationes circa series in hac forma generalis
>
> $$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

Euler used $\int\frac{1}{z^m}$ to denote what we now write as $z(m)$. He writes P for $z*(m,n)$ and Q for $z*(n,m)$. In Euler's notation, and in the original 1776 publication, he wrote his version of formula 3 like this:

> $$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}=P$$
> $$1+\frac{1}{2^n}\left(1+\frac{1}{2^m}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}\right)+\text{etc.}=Q$$
>
> ex principio supra stabilito habebimus
>
> $$P+Q=\int\frac{1}{z^m}\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}$$

A few paragraphs later, Euler improves his notation, and denotes $z*(m,n)$ by $\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)$ instead of by P. Then formula 3 becomes

$$\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int\frac{1}{z^n}\left(\frac{1}{y^n}\right)=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}$$

Later in the paper, Euler develops some general results about multi-zeta values, especially those that he wrote as $\int\frac{1}{z^n}\left(\frac{1}{y}\right)$ and we would write as $z*(n,1)$. He shows easily that

$$\int\frac{1}{z^2}\left(\frac{1}{y}\right)=2\int\frac{1}{z^3}\text{, that is } z*(2,1)=2z(3).$$

It takes a bit more work for him to show that

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right) = \frac{3}{2}\int \frac{1}{z^2}\cdot\int \frac{1}{z^2} - \frac{5}{2}\int \frac{1}{z^4}\text{, that is } z\,*(3,1) = \frac{3}{2}z\,(2)^2 - \frac{5}{2}z\,(4),$$

and then still more work to find that

$$\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^5} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^3}\text{, that is } z\,*(3,1) = 3z\,(5) - z\,(2)\cdot z\,(3).$$

Being Euler, he continues for more than 12 pages, stopping with

$$\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\int \frac{1}{z^8} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^7} + 3\int \frac{1}{z^4}\cdot\int \frac{1}{z^6} - \frac{1}{2}\int \frac{1}{z^5}\cdot\int \frac{1}{z^5} - \frac{11}{2}\int \frac{1}{z^{10}}.$$

We won't translate this into modern notation. In this work, the patterns are not evident, but he applies formula 3 several times and transforms the results into

$$2\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^3}$$
$$2\int \frac{1}{z^3}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^4} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^2}$$
$$2\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^6} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^3}$$

and finally

$$2\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 11\int \frac{1}{z^{10}} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^8} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^7} - 2\int \frac{1}{z^4}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^5}\cdot\int \frac{1}{z^5}.$$

From this, the pattern is evident. In modern notation, it reads

$$2z\,*(n,1) = (n+2)\,z\,(n+1) - \sum_{i=1}^{n-2} z\,(n-i)\,z\,(i+1)$$

This is a form of what is now known as Euler's decomposition formula for the double zeta function, and more than two centuries later it is still an interesting result,

Special thanks to Michael E. Hoffman for inspiring and helping with this column.

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