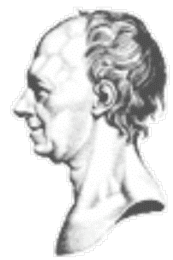

How Euler Did It

by Ed Sandifer

A theorem of Newton

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Early in our algebra careers we learn the basic relationship between the coefficients of a monic quadratic polynomial and the roots of that polynomial. If the roots are α and β and if the polynomial is $x^2 - Ax + B$, then $A = \alpha + \beta$ and $B = \alpha\beta$. Not too long afterwards, we learn that this fact generalizes to higher degree polynomials. As Euler said it, if a polynomial

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \cdots \pm N = 0$$

has roots $\alpha, \beta, \gamma, \delta, \ldots \nu$, then

A = sum of all the roots $= \alpha + \beta + \gamma + \delta + \ldots + \nu$,
B = sum of products taken two at a time $= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma +$ etc.
C = sum of products taken three at a time $= \alpha\beta\gamma +$ etc.
D = sum of products taken four at a time $= \alpha\beta\gamma\delta +$ etc.
etc., and
N = product of all roots $= \alpha\beta\gamma\delta \ldots \nu$.

These facts are very well known, and Euler has no interest in proving them.

Then there are so many other things to learn that most of us don't learn a closely related system of equations that tells us about the sum of the *powers* of the roots. Indeed, Euler writes the sum of the powers of the roots of the polynomial using the notation

$$\int \alpha = \alpha + \beta + \gamma + \cdots + \nu,$$

$$\int \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \cdots + \nu^2,$$

$$\int \alpha^3 = \alpha^3 + \beta^3 + \gamma^3 + \cdots + \nu^3,$$

etc.

This overworks the symbol α, making it both a particular root and at the same time a representative of all the other roots. He also uses the integral sign, $\int$, in one of its 18th century senses, as a summation sign.

With this notation in place, we can state the closely related system of equations we mentioned above. Euler wrote it as

$$\begin{aligned}
&\int \alpha = A, \\
&\int \alpha^2 = A\int \alpha - 2B, \\
&\int \alpha^3 = A\int \alpha^2 - B\int \alpha + 3C, \\
&\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D, \\
&\int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E, \\
&\int \alpha^6 = A\int \alpha^5 - B\int \alpha^4 + C\int \alpha^3 - D\int \alpha^2 + E\int \alpha - 6F, \\
&\qquad \text{etc.}
\end{aligned}$$

Euler attributes these equations to Newton, apparently referring to his *Arithmetica universalis* of 1707. The Editors of Euler's *Opera omnia* Series I volume 6, Ferdinand Rudio, Adolf Krazer and Paul Stäckel, cite evidence that the formulas had been known earlier to Girard in 1629 and to Leibniz "certainly not after 1678."

Let's make sure that we know what Euler means by considering an example. The polynomial $x^4 - 10x^3 + 35x^2 - 50x + 24$ has roots 1, 2, 3 and 4. Indeed, it is easy to check two of the coefficients, $A = 1 + 2 + 3 + 4$ and $D = 1 \cdot 2 \cdot 3 \cdot 4$. The other two are a bit more tedious, but

$$\begin{aligned}
&35 = 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4, \quad \text{and} \\
&50 = 1\cdot 2\cdot 3 + 1\cdot 2\cdot 4 + 1\cdot 3\cdot 4 + 2\cdot 3\cdot 4.
\end{aligned}$$

Also, using Euler's notation for the sums of the powers,

$$\begin{aligned}
&\int \alpha = 1 + 2 + 3 + 4 = 10, \\
&\int \alpha^2 = 1 + 4 + 9 + 16 = 30, \\
&\int \alpha^3 = 1 + 8 + 27 + 64 = 100, \quad \text{and} \\
&\int \alpha^4 = 1 + 16 + 81 + 256 = 354.
\end{aligned}$$

So, Newton's formulas claim that

$$\int \alpha = A = 10,$$

$$
\begin{aligned}
\int \alpha^2 &= A\int \alpha - 2B \\
&= 10\cdot 10 - 2\cdot 35 \\
&= 30, \\
\int \alpha^3 &= A\int \alpha^2 - B\int \alpha + 3C \\
&= 10\cdot 30 - 35\cdot 10 + 3\cdot 50 \\
&= 100, \quad \text{and} \\
\int \alpha^4 &= A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D \\
&= 10\cdot 100 - 35\cdot 30 + 50\cdot 10 - 4\cdot 24 \\
&= 354,
\end{aligned}
$$

as promised.

Euler used these formulas to great effect throughout his career, notably in his solution to the Basel problem, [E41] and several times in his *Introductio in analysin infinitorum* [E101]. In 1747, he decided to prove them. The result was [E153], a short article, 11 pages, with a long title, Demonstratio gemina theorematis neutoniani quo traditur relatio inter coefficientes cuiusvis aequationis algebraicae et summas potestatum radicum eiusdem, "Proof of the basis of a theorem of Newton, which derives a relation between the coefficients of any algebraic equation and the sums of the powers of the roots of that equation," which was published in 1750, and which contains two very different proofs of the result.

He notes that the first equation requires no proof at all, and the second one is quite easy. He writes

$$A^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \text{etc.} + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + \text{etc.}$$

Thus we get

$$A^2 = \int \alpha^2 + 2B$$

and so

$$\int \alpha^2 = A^2 - 2B = A\int \alpha - 2B.$$

Euler claims that he could prove the other formulas similarly, one at a time, but that it would be a great deal of work. Moreover, he writes that others have found these formulas to be most useful, but nobody seems to have proved them "except by induction." By this he means that they have been observed to be true in a great many cases, and never been seen to be false. Still, he thinks that it is so important that they be proved that he offers to do it twice.

Euler's first proof will be based on calculus. Let

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \cdots \pm N = Z$$

Factor Z as

$$Z = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\cdots(x-\nu).$$

Take logarithms and get

$$\ln Z = \ln(x-\alpha) + \ln(x-\beta) + \ln(x-\gamma) + \ln(x-\delta) + \cdots + \ln(x-\nu).$$

The formal manipulations of Euler's time required that he work with differentials instead of derivatives, so Euler takes the differentials,

$$\frac{dZ}{Z} = \frac{dx}{x-\alpha} + \frac{dx}{x-\beta} + \frac{dx}{x-\gamma} + \frac{dx}{x-\delta} + \cdots + \frac{dx}{x-\nu},$$

then he divides by dx to get

$$\frac{dZ}{Zdx} = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \frac{1}{x-\delta} + \cdots + \frac{1}{x-\nu}.$$

He expands each of the quotients on the right as geometric series to get

$$\frac{1}{x-\alpha} = \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \frac{\alpha^4}{x^5} + \frac{\alpha^5}{x^6} + \text{etc.}$$
$$\frac{1}{x-\beta} = \frac{1}{x} + \frac{\beta}{x^2} + \frac{\beta^2}{x^3} + \frac{\beta^3}{x^4} + \frac{\beta^4}{x^5} + \frac{\beta^5}{x^6} + \text{etc.}$$
$$\frac{1}{x-\gamma} = \frac{1}{x} + \frac{\gamma}{x^2} + \frac{\gamma^2}{x^3} + \frac{\gamma^3}{x^4} + \frac{\gamma^4}{x^5} + \frac{\gamma^5}{x^6} + \text{etc.}$$
$$\text{etc.}$$

$$\frac{1}{x-\nu} = \frac{1}{x} + \frac{\nu}{x^2} + \frac{\nu^2}{x^3} + \frac{\nu^3}{x^4} + \frac{\nu^4}{x^5} + \frac{\nu^5}{x^6} + \text{etc.}$$

If we add up these series, collect like powers of x, and use the above definitions of the symbols $\int \alpha$, $\int \alpha^2$, $\int \alpha^3$, etc., we get

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2}\int \alpha + \frac{1}{x^3}\int \alpha^2 + \frac{1}{x^4}\int \alpha^3 + \frac{1}{x^5}\int \alpha^4 + \text{etc.}$$

But, from the definition of Z as a polynomial, we also have

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \text{etc.}$$

and so

$$\frac{dZ}{Zdx} = \frac{nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \text{etc.}}{x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}}.$$

Now we have two expressions for the same quantity, $\frac{dZ}{Zdx}$. Set them equal to each other, and multiply both sides by Z, the polynomial in the denominator of the second expression. We get

$$
\begin{array}{lllllll}
nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} & & & & & & -\text{etc.} \\
= nx^{n-1} & + \; x^{n-2}\int \alpha & + \; x^{n-3}\int \alpha^2 & + \; x^{n-4}\int \alpha^3 & + \; x^{n-5}\int \alpha^4 & & +\text{etc.} \\
 & - \; nAx^{n-2} & - \; Ax^{n-3}\int \alpha & - Ax^{n-4}\int \alpha^2 & - Ax^{n-5}\int \alpha^3 & & -\text{etc.} \\
 & & +nBx^{n-3} & + Bx^{n-4}\int \alpha & + Bx^{n-5}\int \alpha^2 & & +\text{etc.} \\
 & & & -nCx^{n-4} & - Cx^{n-5}\int \alpha & & -\text{etc.} \\
 & & & & +nDx^{n-5} & & +\text{etc.}
\end{array}
$$

Now Euler uses one of his favorite tricks and matches the coefficients of powers of x. For the $(n-1)$st power he gets $n = n$. We knew that, but for the other powers, he gets

$$
\begin{aligned}
-(n-1)A &= \int \alpha - nA, \\
+(n-2)B &= \int \alpha^2 - A\int \alpha + nB, \\
-(n-3)C &= \int \alpha^3 - A\int \alpha^2 + B\int \alpha - nC, \\
+(n-4)D &= \int \alpha^4 - A\int \alpha^3 + B\int \alpha^2 - C\int \alpha + nD \\
&\text{etc.}
\end{aligned}
$$

From these, just a little bit of algebra gives Newton's result,

$$
\begin{aligned}
\int \alpha &= A, \\
\int \alpha^2 &= A\int \alpha - 2B, \\
\int \alpha^3 &= A\int \alpha^2 - B\int \alpha + 3C, \\
\int \alpha^4 &= A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D, \\
\int \alpha^5 &= A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E, \\
&\text{etc.}
\end{aligned}
$$

Euler's second proof is almost completely different, and it relies on manipulations that are seldom found in a modern mathematics curriculum. They were better known a century ago when the editors of the *Opera omnia* were planning the contents of each volume of the series. At the time such techniques were grouped under the heading "Theory of equations." Because of the nature of this second proof, the Editors put E-153 in volume 6 of their first series, the volume titled *Algebraic articles pertaining to the theory of equations*.

To illustrate what he has in mind for his second proof, Euler takes the degree $n = 5$. Each of his steps has obvious analogies for higher degrees, and after showing each step in detail for $n = 5$, he tells us what the corresponding result would be for a general value of n. For $n = 5$, the polynomial equation is

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

and the roots are α, β, γ, δ and ε. If we substitute these roots into the equation, we get the system

$$\begin{aligned}
&\alpha^5 - A\alpha^4 + B\alpha^3 - C\alpha^2 + D\alpha - E = 0,\\
&\beta^5 - A\beta^4 + B\beta^3 - C\beta^2 + D\beta - E = 0,\\
&\gamma^5 - A\gamma^4 + B\gamma^3 - C\gamma^2 + D\gamma - E = 0,\\
&\delta^5 - A\delta^4 + B\delta^3 - C\delta^2 + D\delta - E = 0,\\
&\varepsilon^5 - A\varepsilon^4 + B\varepsilon^3 - C\varepsilon^2 + D\varepsilon - E = 0,
\end{aligned}$$

Sum these, and we get, using the notation above,

$$\int \alpha^5 - A\int \alpha^4 + B\int \alpha^3 - C\int \alpha^2 + D\int \alpha + 5E = 0,$$

so

$$\text{(1)} \qquad \int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E.$$

We will use the nth degree analog of equation (1) later.

Now we form a sequence of polynomials of lower degrees, based on the coefficients of the original polynomial, namely

I.	$x - A = 0,$	and let its root be p;
II.	$x^2 - Ax + B,$	and let one of its roots be q;
III.	$x^3 - Ax^2 + Bx - C,$	and let one of its roots be r; and
IV.	$x^4 - Ax^3 + Bx^2 - Cx + D,$	and let one of its roots be s.

For each of these polynomials, the sum of the roots will be A. For polynomials II, III and IV, the sum of the products of the roots taken two at a time will be B. For III and IV, the sum of the products taken three at a time will be C, and for IV, the product of all four roots will be D.

Now bring the original polynomial equation back into the mix, and we get that

$$(2)\qquad \begin{aligned}
&\int \alpha = \int s = \int r = \int q = \int p,\\
&\int \alpha^2 = \int s^2 = \int r^2 = \int q^2,\\
&\int \alpha^3 = \int s^3 = \int r^3,\\
&\int \alpha^4 = \int s^4.
\end{aligned}$$

Now apply equation (1) to polynomial I to get

$$\int p = A.$$

Likewise, applying it to polynomials II, III an IV, we get

$$\begin{aligned}
&\int q^2 = A\int q - 2B,\\
&\int r^3 = A\int r^2 - B\int r + 3C, \quad \text{and}\\
&\int s^4 = A\int s^3 - B\int s^2 + C\int s - 4D.
\end{aligned}$$

Now, into these equations make the substitutions in equations (2) to get Newton's theorem for $n = 5$:

$$\begin{aligned}
&\int \alpha = A,\\
&\int \alpha^2 = A\int \alpha - 2B,\\
&\int \alpha^3 = A\int \alpha^2 - B\int \alpha + 3C,\\
&\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D,\\
&\int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E.
\end{aligned}$$

Euler's comments along the way make it obvious how this can be extended to polynomials of arbitrary degree.

References:

[E41] Euler, Leonhard, De summis serierum reciprocarum, *Commentarii academiae scientiarum imperialis Petropolitanae* **7** (1734/35) 1740, pp. 123-134. Reprinted in *Opera omnia* I.14, pp. 73-86. Available online at EulerArchive.org.

[E101] Euler, Leohnard, *Introductio in analysin infinitorum*, Bosquet, Lausanne, 1748. Available at EulerArchive.org. English translation by John Blanton, Springer, New York, 1988 and 1990.

[E153] Euler, Leonhard, Demonstratio gemina theorematis neutoniani quo traditur relatio inter coefficients cuiusvis aequationis algebraicae et summas potestatum radicum eiusdem, *Opuscula varii argumenti* **2**, 1750, pp. 108-120. Reprinted in *Opera omnia* I.6, pp. 20-30. The original Latin and an English translation by Jordan Bell are available online at EulerArchive.org.

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