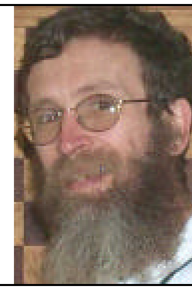
MAA Online

How Euler Did It

by Ed Sandifer

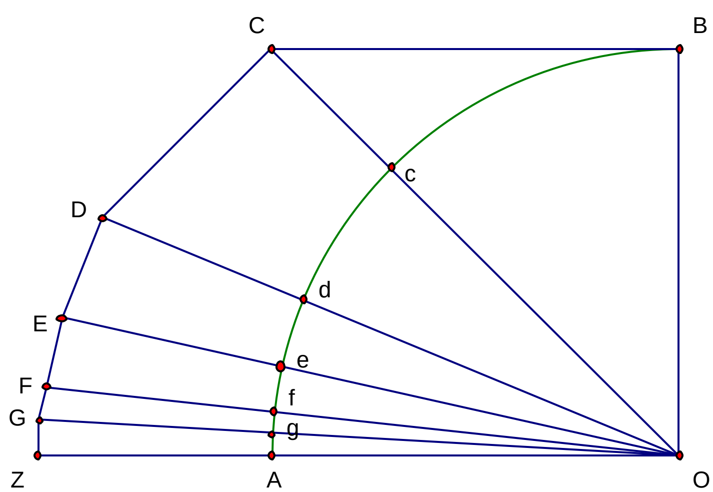


A Product of Secants

May 2008

When an interesting illustration catches our eye, we sometimes stop to figure out what it is. But when I first saw this illustration I was in a hurry. I resolved to come back to it "later." Now that later has finally arrived, I'm glad I remembered to go back.

The picture that caught my eye was the squarish-looking spiral below. It was part of the *Summarium* of [E275], "Notes on a certain passage of Descartes for looking at the quadrature of the circle." The *Summarium* is a summary of an article, usually written by the editor of the journal, that is printed at the beginning of the volume. This time, the *Summarium* was four pages long, and the article itself was twelve.



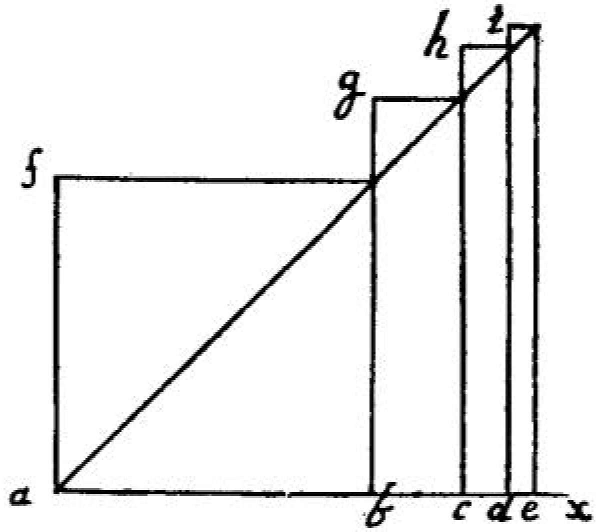
The *Summarium* gives us a bit of history that is not included in the article itself. The Editor tells us that "the circumference of a circle is incommensurable with its diameter," or, as we would say it now, p is an irrational number. He goes on to tell us that Archimedes approximated the ratio as 7 to 22 and Metius gave us 113 to 355.

A bit later, and with only this discussion of Archimedes and Metius as motivation, the Editor asks us to let q be "the length of the quadrant of a circle whose radius is equal to 1," what we would denote $\frac{p}{4}$. Then

$$q = \sec\frac{1}{2}q \cdot \sec\frac{1}{4}q \cdot \sec\frac{1}{8}q \cdot \sec\frac{1}{16}q \cdot \sec\frac{1}{32}q \cdot \text{etc.}$$

A minute with Maple® confirms this, at least to ten decimal places, and the Editor leads us to believe that the illustration should help to convince us that it is true. There is no mention of Descartes in the *Summarium*.

Euler begins the article itself describing a very different construction and with a different illustration. He tells us that the method is due to Descartes and that it "indicates brilliantly the insightful character of its discoverer."[1] As we go through Descartes' construction, it is helpful to note that Descartes describes a rectangle or a square by telling us two diagonally opposite corners of the shape. So, in the figure below, he calls the large square *bf*, and the rectangle next to it is *cg*.



Using the figure above, Descartes gives a procedure that begins with the length *ab* and the square on that length, *bf*. Then he constructs a new length *ax*. He claims (and Euler agrees) that the length *ax* forms the diameter of a circle that has the same circumference as the square *bf*. Hence, if $ab = 1$, then$ax = \frac{4}{p}$, about 1.2732. Here's how the construction works.

Take *ao* to be the ray containing the diagonal of square *bf*. Beside this, construct rectangle *cg* so that its area is ¼ the area of square *bf*, and so that its (unnamed) corner lies on the ray *ao*.

Beside this, construct another rectangle *dh*, with its area ¼ the area of rectangle *bf*, and again with its corner on the ray *ao*.

[1] Here and elsewhere, when we quote from the text of E275, we usually follow the translation of Jordan Bell, available at EulerArchive.org and at the arXiv. Thank you, Jordan, for your many fine translations of Euler's work.

Continue constructing rectangles, each with ¼ the area of the previous one, and each with its corner on the ray. It is easy to see that the sum of the bases of these rectangles,

$$ab + bc + cd + de + \cdots$$

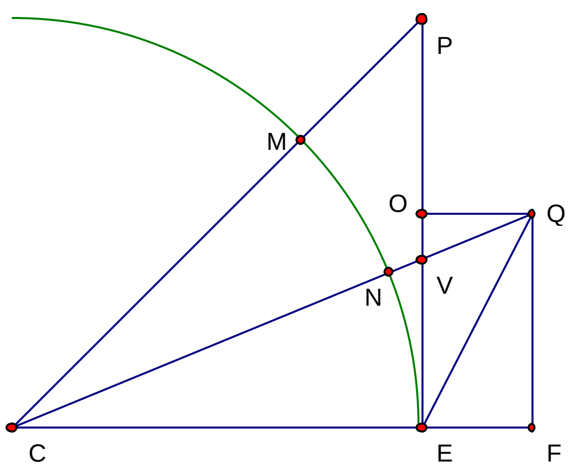
converges to some length, call it ax, but it is not so easy to see how ax is related to ab, or how this has anything to do with the circumference of a circle. Descartes, in the style of his times, doesn't tell us. Euler, though, sets out to prove it, and he shares the details of his proof with us.

But first, Euler proposes to solve the following:

PROBLEM

> *Given a circle around which a regular polygon is circumscribed, to find another circle, about which if a regular polygon with twice as many sides is circumscribed, the perimeter of the first polygon will be equal to the perimeter of the second one.*

It is not yet clear what this has to do with Descartes' construction, or with the product of secants and the squarish spiral we saw in the *Summarium*, but those of us who remember how Archimedes approximated p as 22/7 will recognize how this problem is related to the value of p. Euler uses the figure below to solve his problem.



Here, MNE is an arc of the first circle. That circle has center C and radius CE. The segment PE is half of one side of the first polygon, circumscribed about MNE.

The new circle will be larger than the original one, so if we let CF be the radius of the new circle, we will have CE < CF.

Let FQ be a half-side of the new polygon. Since the new polygon has the same circumference, but twice as many sides as the original, we have EP = 2 FQ. Likewise, $\angle$ECP = 2 $\angle$FCQ.

Let O be the midpoint of PE. Then QO || EF. Also, let V be the point where the radius CQ intersects PE. Note that V is between O and E, that is to say, EV < EO = ½ PE.

Now Euler leads us through some triangle geometry. Some steps are easy, but one involves knowing something that is largely forgotten today.

First, EV : CE = FQ : CF because ?CEV ˜ ?CFQ. That was easy.

Second, EV : CE = EP : CE + CP. This is not so easy. Here's how I figured it out. I compared ?PCE with ?VCE. Recall that ∠VCE = ½ ∠PCE. I saw that the ratios CE : EV, CE : EP and CP : EP involved cotangents and cosecants of these angles, so I looked up the half-angle formula for cotangent and found it to be

$$\cot\frac{q}{2} = \cot q + \sec q.$$

I think it was the first time in my life that I'd ever *used* the half angle formula for cotangents, but I thank my high school geometry teacher, Ken Solem, for teaching me that there is one, so I'd know to look it up when I needed it. There may be a simpler way, but this was quick.

Third, by combining the first two steps, we get FQ : CF = EP : CE + CP.

Because FQ = ½ CF, this last proportion tells us that

$$CF = \frac{1}{2}(CE + CP).$$

Subtracting CE from both sides gives

$$EF = \frac{1}{2}(CP - CE)'$$

Multiplying these last two together gives

$$\begin{aligned} CF \cdot EF &= \frac{1}{4}\left(CP^2 - CE^2\right) \\ &= \frac{1}{4}EP^2. \end{aligned}$$

The first equality is just algebra and the second line is an application of the Pythagorean theorem to the right triangle CEP.

This solves Euler's problem because the point F is now defined so that the rectangle with sides CF and EF has as its area one-fourth the area of the rectangle with sides EP and FQ. That, in turn, equals the area of the square with sides FQ.

This result is a little awkward to use, so Euler "cleans it up" with four corollaries:

Corollary 1: Because $CF \cdot EF = FQ^2$, we have $CF : FQ = FQ : EF$. This means we have similar triangles CFQ ˜ FQE, so that ∠FCQ = ∠FQE.

Corollary 2: From Corollary 1, CE : EV = EO : EF, so that point F can be defined by drawing from the point O a straight line perpendicular to the line CV extended, and finding where that new line intersects the base line CE.

Corollary 3: If the polygon circumscribing circle ENM has n sides, then $\angle ECP = \frac{p}{n}$, and $\angle FCQ = \frac{p}{2n}$. If we let the radius CE = r, then

$$EP = r\tan\frac{p}{n} \quad \text{and} \quad FQ = \frac{1}{2}r\tan\frac{p}{n}.$$

Corollary 4: Because $\angle FQE = \frac{p}{2n}$,

$$EF = FQ\tan\frac{p}{2n} = \frac{1}{2}r\tan\frac{p}{n}\tan\frac{p}{2n}.$$

If we let CF = s, then we have

$$FQ = s\tan\frac{p}{2n},$$

and because

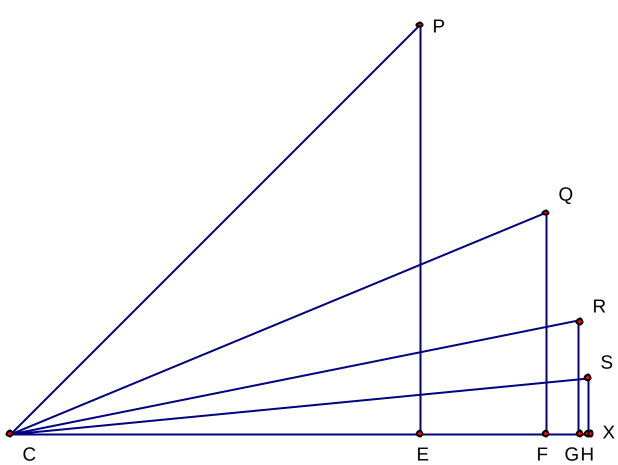
$$FQ = \frac{1}{2}r\tan\frac{p}{n},$$

we have

$$s = \frac{1}{2}r\tan\frac{p}{n}\cot\frac{p}{2n}.$$

Thus, we have a direct means of finding the length CF from the original length CE and the number of sides n.

Now Euler is ready to prove that Descartes' construction does what is claimed. This requires a new figure:



Here, we let CE be the radius of a circle inscribed in a square, CF that of an octagon, CG of a hexadecagon, CH, etc., and let EP, FQ, GR, HS be the corresponding half-sides. As we saw before,

$$FQ = \frac{1}{2}EP, \quad GR = \frac{1}{2}FQ = \frac{1}{4}EP, \quad HS = \frac{1}{2}GR = \frac{1}{4}FQ = \frac{1}{8}EP, \quad \text{etc.}$$

From the preceding problem,

$$CF \cdot EF = \frac{1}{4}EP^2 = FQ^2.$$

Similarly,

$$CG \cdot FG = \frac{1}{4}FQ^2 = \frac{1}{4}CF \cdot EF = GR^2,$$
$$CH \cdot GH = \frac{1}{4}GR^2 = \frac{1}{4}CG \cdot FG = HS^2, \quad \text{etc.}$$

With the points F, G, H, etc. determined in this way, we get Descartes' construction. Moreover, the points E, F, G, H, etc. "ultimately approach" the point x, the radius Cx will be the radius of the circle the circumference of which is approached by the corresponding polygons.

Thus, the construction of Descartes is proved. The construction leads to a means of approximating p that Euler describes in another corollary:

Corollary 1: If we take CE = a, CF = b, CG = c, CH = d, etc., we have EP = a, and we get the recursive sequence

$$b(b-a) = \frac{1}{4}aa, \quad c(c-b) = \frac{1}{4}b(b-a), \quad d(d-c) = \frac{1}{4}c(c-b), \quad \text{etc.}$$

From this, quadratic formula gives us

$$b = \frac{a+\sqrt{2aa}}{2}, \quad c = \frac{b+\sqrt{2bb-ab}}{2}, \quad d = \frac{c+\sqrt{2cc-bc}}{2}, \quad \text{etc.}$$

and these quantities, taken to infinity, give the radius of the circle with perimeter equal to 8a.

Indeed, if we take a = 1, then the first several values of this sequence are

$$\begin{aligned} a &= 1.00000 \\ b &= 1.20711 \\ c &= 1.25683 \\ d &= 1.26915 \\ e &= 1.27222 \\ f &= 1.27298 \\ g &= 1.27318 \end{aligned}$$

and these seem to be converging towards the required value of 4/p ˜ 1.27323954. Indeed, they agree to these eight decimal places on the 14th step (taking a = 1 as step 1).

Let us pause to take stock of what has happened so far in this article. Its title promised that we would learn about a method of Descartes for approximating p. We have done that. However, the *Summarium*, as well as the title I chose for the column, advertised an infinite product of secants. We haven't seen such an infinite product, nor have we seen anything of that spiral illustration that caught my eye in the first place. It's time to see what we can do about that. Euler begins a new problem.

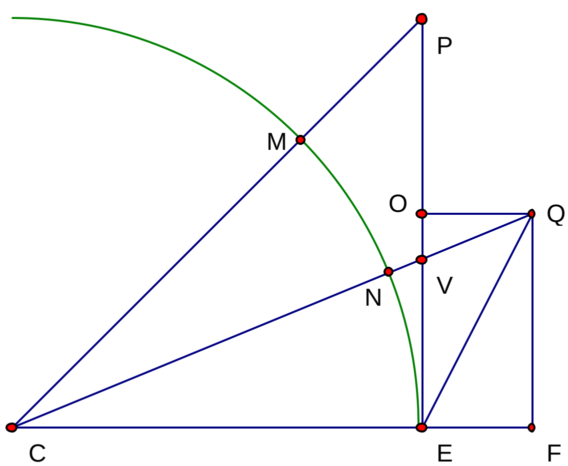
PROBLEM

Taking j *to be any arc of a circle of radius* 1*, to find the sum of the infinite series*

$$\tan j + \frac{1}{2}\tan\frac{1}{2}j + \frac{1}{4}\tan\frac{1}{4}j + \frac{1}{8}\tan\frac{1}{8}j + \frac{1}{16}\tan\frac{1}{16}j + \text{etc.}$$

To solve this problem, Euler brings back the figure from his first problem. This time he lets $\angle ECP = j$ be any angle, and $\angle FCQ = \frac{1}{2}j$. He scales his drawing so that FQ = 1, which makes EP = 2. Then $CE = 2\cot j$, $CF = \cot\frac{1}{2}j$ and $EF = \tan\frac{1}{2}j$. This last formula requires that we recall from Corollary 1 of the first problem that ? FQE ˜ ? FCQ. Now CE = CF – EF, so

$$2\cot j = \cot\tfrac{1}{2}j - \tan\tfrac{1}{2}j \quad \text{and} \quad \tan\tfrac{1}{2}j = \cot\tfrac{1}{2}j - 2\cot j.$$



In the same way, $\tan j = \cot j - 2\cot 2j$. We can apply these identities to get every term of the series in the problem, and we find that

$$\begin{aligned}
\tan j &= \cot j - 2\cot 2j, \\
\tfrac{1}{2}\tan\tfrac{1}{2}j &= \tfrac{1}{2}\cot\tfrac{1}{2}j - \cot j, \\
\tfrac{1}{4}\tan\tfrac{1}{4}j &= \tfrac{1}{4}\cot\tfrac{1}{4}j - \tfrac{1}{2}\cot\tfrac{1}{2}j, \\
\tfrac{1}{8}\tan\tfrac{1}{8}j &= \tfrac{1}{8}\cot\tfrac{1}{8}j - \tfrac{1}{4}\cot\tfrac{1}{4}j, \\
&\textit{etc.}
\end{aligned}$$

Bravely adding both sides of these together, and in characteristic Eulerian form, taking *n* to be an infinite number, we see that on the left we get exactly the series we are trying to sum, and on the right a riot of cancellation from which the only terms that survive are

$$-2\cot 2j + \frac{1}{n}\cot\frac{1}{n}j .$$

The second term is subject to l'Hôpital's rule, and becomes just $\frac{1}{j}$, so the sum of Euler's series and the solution to the latest problem is

$$\frac{1}{j} - 2\cot 2j .$$

From here, Euler gives a few different paths to his product of secants. We'll describe my favorite. Start with

$$\tan j + \frac{1}{2}\tan\frac{1}{2}j + \frac{1}{4}\tan\frac{1}{4}j + \frac{1}{8}\tan\frac{1}{8}j + + \frac{1}{16}\tan\frac{1}{16}j + \mathsf{L} = \frac{1}{j} - 2\cot 2j .$$

Integrate both sides to get

$$-\ln\cos j - \ln\cos\frac{1}{2}j - \ln\cos\frac{1}{4}j - \ln\cos\frac{1}{8}j - \ln\cos\frac{1}{16}j - \mathsf{L} = \ln j - \ln\sin 2j + \text{Const.}$$

Taking $j = 0$ leads to finding that the constant is $\ln 2$. Also, $-\ln\cos q = \ln\sec q$, so, by the laws of logarithms we get

$$\frac{1}{\cos j \cdot \cos\frac{1}{2}j \cdot \cos\frac{1}{4}j \cdot \cos\frac{1}{8}j \cdot \cos\frac{1}{16}j \cdots} = \frac{2j}{\sin 2j},$$

or, what amounts to the same thing,

$$\sec j \cdot \sec\frac{1}{2}j \cdot \sec\frac{1}{4}j \cdot \sec\frac{1}{8}j \cdot \sec\frac{1}{16}j \cdots = \frac{2j}{\sin 2j}.$$

The product given in the *Summarium* is the special case of this formula where $j = \frac{p}{4}$.

And what does Euler say of the pretty spiral that started it all? Nothing. He leaves that to us. Take AB = OB = 1. Then we might begin by noting that ?OBC is a right triangle and $\angle BOC = \frac{p}{4}$. So, $\frac{OC}{OB} = \sec\frac{p}{4}$. Since OB = 1, this makes $OC = \sec\frac{p}{4}$.

Then ?OCD is a right triangle and $\angle COD = \frac{p}{8}$. So $\frac{OD}{OC} = \sec\frac{p}{8}$. We know OC from the previous step. The result follows by repeating this process infinitely many times.

So, a pretty picture leads to a pleasing result.

References:

[E275] Euler, Leonhard, Annotiationes in locum quondam Cartesii ad circuli quadraturam spectantem, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **8** (1760/61) 1763, pp. 24-27, 157-168. Reprinted in *Opera omnia* I.15 pp. 1-15. Available online, along with an English translation of the article (not the *Summarium*) by Jordan Bell, at EulerArchive.org.

[Descartes] Descartes, René, "Circuli Quadratio," *Oeuvres de Descartes*, ed. Adams, C. and P. Tannery, Léopold Cerf, Paris, 1908, v. 10, pp. 304-305.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 36 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org). His first book, *The Early Mathematics of Leonhard Euler*, was published by the MAA in December 2006, as part of the celebrations of Euler's tercentennial in 2007. The MAA published a collection of forty *How Euler Did It* columns in June 2007.

How Euler Did It is updated each month.