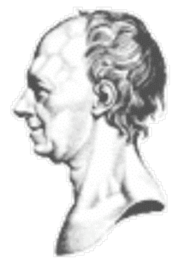

How Euler Did It

by Ed Sandifer

The Euler line

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A hundred years ago, if you'd asked people why Leonhard Euler was famous, those who had an answer would very likely have mentioned his discovery of the Euler line, the remarkable property that the orthocenter, the center of gravity and the circumcenter of a triangle are collinear. But times change, and so do fashions and the standards by which we interpret history.

At the end of the nineteenth century, triangle geometry was regarded as one of the crowning achievements of mathematics, and the Euler line was one of its finest jewels. Mathematicians who neglected triangle geometry to study exotic new fields like logic, abstract algebra or topology were taking brave risks to their professional careers. Now it would be the aspiring triangle geometer taking the risks.

Still, the late H. S. M. Coxeter made a long and distinguished career without straying far from the world of triangles, and he introduced hundreds of others to their delightful properties, especially the Euler line. This month, we look at how Euler discovered the Euler line and what he was trying to do when he discovered it. We will find that the discovery was rather incidental to the problem he was trying to solve, and that the problem itself was otherwise rather unimportant.

This brings us to the 325th paper in Gustav Eneström's index of Euler's published work, "Solutio facilis problematum quorumdam geometricorum difficillimorum" (An easy solution to a very difficult problem in geometry) [E325].   Euler wrote the paper in 1763 when he lived in Berlin and worked at the academy of Frederick the Great. The Seven Years War, which extended from 1756 to 1763, was just ending. Late in the war, Berlin had been occupied by foreign troops. Euler and the other academicians had lived the last years in fear for their own safety and that of their families, but a dramatic turn of events enabled Frederic to snatch victory from the jaws of defeat and win the war. When he returned to Berlin, he tried to manage his Academy of Sciences the same way he had managed his troops. In just three years, he had completely alienated Euler and Euler left for St. Petersburg, Russia to work at the academy of Catherine the Great. The paper was published in 1767 in the journal of the St. Petersburg Academy.

Euler begins his paper by reminding us that a triangle has four particularly interesting points:

1. the intersection of its three altitudes, which he denotes by *E*. Since about 1870, people have called this point the *orthocenter* and before that it was called the *Archimedian point*. Euler does not use either term;
2. the intersection of its median lines. Euler labels this point *F* and, as we do today, calls it the center of gravity;
3. the intersection of its angle bisectors. Euler labels this point *G* and calls it the center of the inscribed circle. Since about 1890, people have been calling this point the *incenter*;
4. the intersection of the perpendicular bisectors of the sides. Euler labels this point *H* and notes that it is the center of the circumscribed circle. Since about 1890, people have called it the *circumcenter*.

Modern texts usually use different letters to denote these same points, but as usual we will follow Euler's notation.

Then he announces what he regards as the main results of this paper: If these four points do not coincide, then the triangle is determined. If any two coincide, then all four coincide, and the triangle is equilateral, but it could be any size.

To prepare for his analysis, he defines some notation. He calls his triangle *ABC*, and lets its sides be of lengths *a, b* and *c*, where the side of length *a* is opposite vertex *A*, etc. Euler also denotes the area of the triangle by *A*, and trusts the reader to keep track of whether he is talking about the point *A* or the area *A*. He knows Heron's formula, though he doesn't know it by that name. Here, as in [E135], it is just a formula that he assumes we know about. He gives it in two forms:

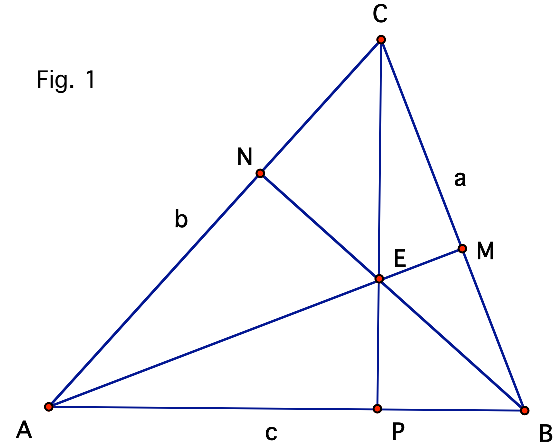
$$AA = \tfrac{1}{16}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$= \tfrac{1}{16}\left(2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4\right),$$

where, if you've been keeping track like we told you to, you know that *AA* denotes the square of the area of triangle *ABC*.

With the notation established, Euler sets out to give the locations of each of the centers, *E, F, G* and *H*, in terms of the lengths of the sides, *a, b* and *c*, and relative to the point *A* as an origin and the side *AB* as an axis. He begins with the orthocenter, *E*.

Let *P* be the point where the line through *C* perpendicular to *AB* intersects *AB* (see Fig. 1). Then *AP* serves as a kind of *x*-coordinate of the point *E*, and *EP* acts as a *y*-coordinate. Likewise, he takes *MA* to be the perpendicular to *BC* and *NB* to be the perpendicular to *AC*, but these lines do not play a role as coordinates.

Fig. 1



Euler tells us that

$$AP = \frac{cc + bb - aa}{2c}.$$

He gives us no reason, but it is easy algebra if you use the law of cosines to write $a^2 = b^2 + c^2 - 2bc\cos A$ and observe that $\cos A = \dfrac{AP}{b}$. That takes care of the *x*-coordinate of the point *E*.

Likewise, $BM = \dfrac{aa + cc - bb}{2a}$. Also, the triangle area formula gives us $A = \tfrac{1}{2} AM \cdot BC$, so that $AM = \dfrac{2A}{a}$. Triangles *ABM* and *AEP* are similar (because they are both right triangles and they share angle *B*), so $AM : BM = AP : EP$, and this easily leads to the second coordinate of the point *E*, namely

$$EP = \frac{(cc + bb - aa)(aa + cc - bb)}{8cA}.$$

Euler repeats similar analysis for each of the other centers. He introduces the points *Q, R* and *S* as the points on side *AB* corresponding to the *x*-coordinates of the centers *F, G* and *H* respectively. For the coordinates of the center of gravity, *F*, he finds

$$AQ = \frac{3cc + bb - aa}{6c} \quad \text{and} \quad QF = \frac{2A}{3c}.$$

For the coordinates of *G*, the center of the inscribed circle, he gets

$$AR = \frac{c + b - a}{2} \quad \text{and} \quad RG = \frac{2A}{a + b + c}.$$

Finally, for the *H*, the center of the circumscribed circle, he finds

$$AS = \tfrac{1}{2} c \quad \text{and} \quad SH = \frac{c(aa + bb - cc)}{8A}.$$

We leave to the reader the pleasant task of checking these calculations. Some are trickier than others.

This concludes the first part of Euler's paper. He has located his four centers in terms of the lengths of the three sides of the triangle. This has taken him about five pages of this 21-page paper.

There are six pairwise distances among these points:

$$\begin{aligned}
EF^2 &= (AP - AQ)^2 + (PE + QF)^2 \\
EG^2 &= (AP - AR)^2 + (PE - RG)^2 \\
EH^2 &= (AP - AS)^2 + (PE - SH)^2 \\
FG^2 &= (AQ - AR)^2 + (QF - RG)^2 \\
FH^2 &= (AQ - AS)^2 + (QF - SH)^2 \\
GH^2 &= (AR - AS)^2 + (RG - SH)^2,
\end{aligned}$$

where EF^2 denotes the square of the length of segment *EF*, etc.

To investigate these distances, it will be convenient to take

$$a+b+c=p, \quad ab+ac+bc=q \quad \text{and} \quad abc=r.$$

Later it will be important that this definition of *p, q, r* makes the lengths of the sides, *a, b* and *c*, equal to the roots of the cubic equation

$$z^3 - pzz + qz - r = 0.$$

Then it will also be useful to know that

$$\begin{aligned} aa+bb+cc &= pp-2q \\ aabb+aacc+bbcc &= qq-2pr \\ a^4+b^4+c^4 &= p^4-4ppq+2qq+4pr \end{aligned}$$

and that the area A can be expressed as

$$AA = \tfrac{1}{16}p\left(-p^3+4pq-8r\right) = \frac{-p^4+4ppq-8pr}{16}.$$

Six pages of rather tedious and straightforward calculations lead Euler to relations among the six distances between pairs of these four points. Referring to his Fig. 5, he eventually gets

I. $EF^2 = \dfrac{rr}{4AA} - \dfrac{4}{9}(pp-2q)$

II. $EG^2 = \dfrac{rr}{4AA} - pp + 3q - \dfrac{4r}{p}$

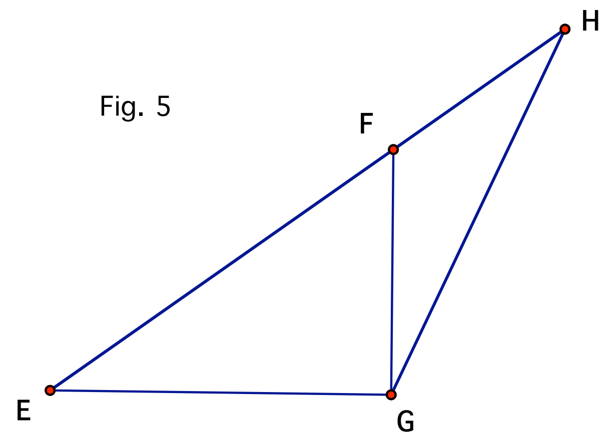
III. $EH^2 = \dfrac{9rr}{16AA} - pp + 2q$

IV. $FG^2 = -\dfrac{1}{9}pp + \dfrac{5}{9}q - \dfrac{2r}{p}$

V. $FH^2 = \dfrac{rr}{16AA} - \dfrac{1}{9}(pp-2q)$

VI. $GH^2 = \dfrac{rr}{16AA} - \dfrac{r}{p}$

Fig. 5



This looks like just another list of formulas, but there is a gem hidden here. Euler sees that $EH = \tfrac{3}{2}EF$ and $FH = \tfrac{1}{2}EF$. He remarks that this implies that if the points *E* and *F* are known, then the point *H* can be found on the straight line through *E* and *F*. He does not specifically mention that because *EF* + *FH* = *EH*, the three points are collinear.

Nothing in Euler's presentation suggests that he thought this was very important or even very interesting. He only mentions that he can find *H* given *E* and *F*, and not that *E* or *F* could be found knowing the other two.

In more modern terms, and with modern emphasis, we give this result by saying that the orthocenter *E*, the center of gravity *F* and the circumcenter *H* are collinear, with $EH = \frac{3}{2}EF$, and we call the line through the three points the Euler line. Moreover, Euler seems almost equally interested in another harder-to-see and surely less important consequence of the same equations, that

$$4GH^2 + 2EG^2 = 3EF^2 + 6FG^2 .$$

But Euler doesn't dwell on this. His problem is not to discover the properties of these various centers of the triangle, but to try to reconstruct the triangle given these centers. Towards this end, he introduces three new values, *P, Q* and *R*, defined in terms of *p, q* and *r* by

$$\frac{rr}{ps} = R, \qquad \frac{r}{p} = Q \qquad \text{and} \qquad pp = P .$$

Then he rewrites the relations given in formulas I to VI in terms of *P, Q* and *R*. He only ends up using three of these formulas:

I. $GH^2 = \frac{1}{4}R - Q$

II. $FH^2 = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2QQ}{9R}$

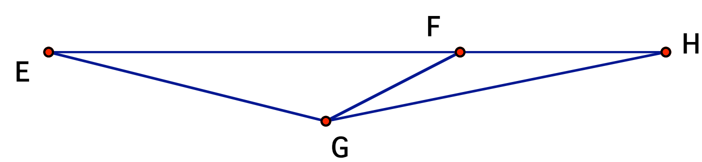
III. $FG^2 = \frac{1}{36}P - \frac{8}{9}Q + \frac{5QQ}{9R}$

Finally, Euler is ready to state and solve the problem that is his reason for writing this paper:

PROBLEM: Given these four points related to a triangle, to construct the triangle.

For reasons he doesn't make exactly clear, Euler divides the problem into two cases. The first case is where the point *G* does not lie on the Euler line, or as Euler says it, the case where the points *F, G* and *H* form a triangle, as shown in his Fig. 6. In the second case, all four lines are on the same line.

Fig. 6



Euler begins by taking

$$GH = f, \qquad FH = g \qquad \text{and} \qquad FG = h.$$

Then, from the statement, such as it is, of the Euler line theorem, and from the remark that follows it, we get

$$EF = 2g, \qquad EH = 3g \qquad \text{and} \qquad EG = \sqrt{6gg + 3hh - 2ff} ,$$

and formulas I, II and III can be rewritten as

I. $ff = \frac{1}{4}R - Q$

II. $gg = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2QQ}{9R}$

III. $hh = \frac{1}{36}P - \frac{8}{9}Q + \frac{5QQ}{9R}$

Solving these for *P, Q* and *R* gives

$$R = \frac{4f^4}{3gg + 6hh - 2ff}, \qquad Q = \frac{3ff(ff - gg - 2hh)}{3gg + 6hh - 2ff}, \qquad P = \frac{27f^4}{3gg + 6hh - 2ff} - 12ff - 15gg + 6hh,$$

and these make

$$\frac{QQ}{R} = \frac{9(ff - gg - 2hh)^2}{4(3gg + 6hh - 2ff)}.$$

Now he writes *p, q* and *r* in terms of *P, Q* and *R* (which means that they are also known in terms of *f, g* and *h*) and gets

$$p = \sqrt{P}, \qquad q = \tfrac{1}{4}P + 2Q + \frac{QQ}{R} \quad \text{and} \quad r = Q\sqrt{P}.$$

Finally, he reminds us that the three sides of the triangle are the three roots of the cubic equation

$$z^3 - pzz + qz - r = 0.$$

In case we're not sure how to use Euler's solution to solve the problem, he does an example. First, like a good teacher, he designs the problem so it will have an easy answer. He considers a triangle with sides $a = 5$, $b = 6$ and $c = 7$. In a sense, this is the simplest acute scalene triangle. He uses the first version of his formulas I to VI to find that for this triangle,

$$ff = \frac{35}{32}, \qquad gg = \frac{155}{288} \quad \text{and} \quad hh = \frac{1}{9}.$$

Now he pretends he doesn't know *a, b* and *c* and that he's only given *ff, gg* and *hh*. The formulas for *P, Q* and *R* in terms of *f, g* and *h* give

$$R = \frac{1225}{24}, \qquad Q = \frac{35}{3}, \qquad P = 324 \quad \text{and} \quad \frac{QQ}{R} = \frac{24}{9} = \frac{8}{3}.$$

These give *p, q* and *r* as

$$p = \sqrt{P} = 18, \qquad q = 107 \quad \text{and} \quad r = \tfrac{35}{3} \cdot 18 = 5 \cdot 6 \cdot 7 = 210.$$

The cubic equation is then

$$z^3 - 18zz + 107z - 210 = 0 .$$

As expected, the three roots of this equation are 5, 6 and 7.

Euler considers the case separately where all four centers lie on one line. He finds that the cubic has a double root and that this gives an isosceles triangle. Perhaps he thought that the slight differences between cubic equations with three distinct roots and those with a double root were enough to merit distinguishing between the cases. He doesn't give any details of why the triangle must be equilateral if all four centers coincide.

In some ways, Euler's discovery of the Euler line is analogous to Columbus's "discovery" of America. Both made their discoveries while looking for something else. Columbus was trying to find China. Euler was trying to find a way to reconstruct a triangle, given the locations of some of its various centers. Neither named his discovery. Columbus never called it "America" and Euler never called it "the Euler line."

Both misunderstood the importance of their discoveries. Columbus believed he had made a great and wonderful discovery, but he thought he's discovered a better route from Europe to the Far East. Euler knew what he'd discovered, but didn't realize how important it would turn out to be.

Finally, Columbus made several more trips to the New World, but Euler, as with his polyhedral formula and the Königsberg bridge problem, made an important discovery but never went back to study it further.

References:

[E135] Euler, Leonhard, Variae demonstrations geometriae, *Novi commentarii academiae scientiarum Petropolitanae* **1** (1747/48) 1750, pp. 38-38, 49-66. Reprinted in *Opera omnia*, Series I vol. 26, pp. 15-32. Also available online, along with an English translation by Adam Glover, at EulerArchive.org.

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