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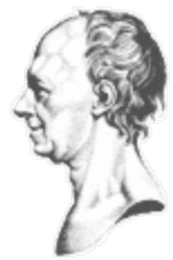

# How Euler Did It

by Ed Sandifer

## Sums (and differences) that are squares

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Late in his life, Euler devoted a lot of time and effort to number theory. Indeed, almost 40% of his papers in number theory were published after he had died. Many of these late papers were on Diophantine equations, usually involving square numbers in one way or another.

Usually, Euler does not tell us what motivates the particular Diophantine equations that he chooses to study. Sometimes he does tell us, as in [E754], Problème de géométrie résolu par l'analyse de Diophante, "A problem in geometry solved by Diophantine analysis". In that paper, Euler seeks three integer numbers, $x$, $y$ and $z$, such that the three quantities

$$\begin{aligned} &2xx + 2yy - zz, \\ &2yy + 2zz - xx \quad \text{and} \\ &2zz + 2xx - yy \end{aligned}$$

are all perfect squares. He explains that $x$, $y$ and $z$ have these properties exactly when a triangle with sides of length $x$, $y$ and $z$ has median lines that have rational length. That paper, E754, was one of the subjects of this column in March 2008. [Sandifer 2008] Thus, the Diophantine equations sometimes hide a geometric motivation.

Other times, though, Euler does not tell us what inspired a particular Diophantine problem, as in [E753], Solutio succincta et elegans problematis quo quaeruntur tres numeri tales ut tam summae quam differentiae binorum sint quadrata, "A succinct and elegant solution to the problem of finding three numbers such that the sum or difference of any two of them is a square number." Here again he seeks three integer numbers, $x$, $y$ and $z$, where $x$ is the largest and $z$ the smallest, this time with the property that all three sums and all three differences,

$$\begin{aligned} &x + y, && x - y, \\ &x + z, && x - z, \\ &y + z \quad \text{and} && y - z, \end{aligned}$$

are perfect squares.

We are left to guess *why* Euler thinks such triples are interesting. Euler hints that he a reason when he opens the paper, "This problem is treated and solved by several authors," but tells us neither who those authors were nor why they were interested. We will return to the question of Euler's motivation at the end of the column.

Before we look at the content of this article, it is probably a good idea to review some of Euler's related works from the 1750s, in particular, [E228], De numeris qui sunt aggregata duorum quadratorum, "On numbers that are the sum of two squares". We will need some lemmas from this article:

Lemma 1: If $2n$ is a sum of two squares, then so also is $n$.

> Proof: Suppose $2n = aa + bb$. Then either $a$ and $b$ are both odd or they are both even. In either case, both $(a + b)/2$ and $(a - b)/2$ are integers, and it is easy to calculate that their squares sum to $n$.

Lemma 2: Suppose that $m$ is the sum of two squares. Then so also is $2m$.

> Proof: Suppose that $m = aa + bb$, with $a$ greater than or equal to $b$. Then $a + b$ and $a - b$ are the numbers whose squares sum to $2m$.

Lemma 3: If a number $n$ is a sum of two squares in two different ways, then $n$ is composite and it can be factored so that each factor is a sum of two squares.

> Proof: The proof is a calculation. If $x = pp + qq = rr + ss$ then $x = (aa + bb)(cc + dd)$ where $p = ac + bd$, $q = ad - bc$, $r = ad + bc$ and $s = ac - bd$.

Returning to the article at hand, we see that Euler proceeds by analysis. He supposes that $x$, $y$ and $z$ are three numbers that satisfy the conditions of the problem, that is their sums and differences are all squares, and he supposes that $x$ is the largest and $z$ is the smallest. Euler claims without proof that there must be integers $p$ and $q$ such that $x = pp + qq$ and $y = 2pq$. This is not obvious, but it is not hard, either. Suppose that $x + y = a^2$ and $x - y = d^2$. Then $x = \frac{a^2 + d^2}{2}$ so, by Lemma 1, $2x$ is the sum of two squares, $p = \frac{a+d}{2}$ and $q = \frac{a-d}{2}$. A little algebra shows that $x = pp + qq$ and $y = 2pq$, as Euler asserted.

Now, because $x = pp + qq$ and $y = 2pq$, we have $x + y = (p+q)^2$ and $x - y = (p-q)^2$, so for that pair, at least, the sum and the difference are squares, as required. Similarly, taking $x = rr + ss$ and $z = 2rs$, we get that $x + z = (r+s)^2$ and $x - z = (r-s)^2$. We are guaranteed from the construction so far that $x + y$, $x - y$, $x + z$ and $x - z$ are all squares. For these four conditions to be satisfied (i.e. $x = x$) we get $rr + ss = pp + qq$. For $x$, $y$ and $z$ to be solutions to the original problem, two more conditions must also be satisfied, that $y + z = 2pq + 2rs$ and that $y - z = 2pq - 2rs$ must also be squares.

Thus Euler has transformed his original problem into a related problem of finding four numbers, $p$, $q$, $r$ and $s$ such that

$x = pp + qq = rr + ss,$
$y + z = 2pq + 2rs$ is a square and
$y - z = 2pq$ is a square.

Then $x$ is composite, and $x = (aa + bb)(cc + dd)$, where $a, b, c$ and $d$ are as given in Lemma 3. Now we can rewrite $y$ and $z$ in terms of $a, b, c$ and $d$, and then find $x + z$ and $x - z$. Because both of these are squares, their product, $yy - zz$ is also a square. This works out to be

$$yy - zz = 16abcd\left(aa - bb\right)\left(dd - cc\right)$$

For this to be a square, the formula

$$ab\left(aa - bb\right)\cdot cd\left(dd - cc\right)$$

must also give a square.

Now, Euler simplifies his problem by discarding some solutions, supposing that $a = d$ and that there exists an integer $n$ such that

$$c(aa - cc) = nnb(aa - bb).$$

He further supposes that $a = c - b,$ perhaps discarding yet more solutions. Then some algebra yields

$$\frac{b}{c} = \frac{nn + 2}{2nn + 1}.$$

Now if we take the obvious possible solution, $b = nn + 2$ and $c = 2nn + 1$, it makes $a = d = nn - 1$, so that the formula $ab(dd - cc)$ reduces to

$$3nn\left(nn - 1\right)\left(nn + 2\right)^2.$$

For this to be a square, $3(nn - 1)$ must also be a square.

This leaves us to find such $n$. Euler introduces yet a new pair of variables, $f$ and $g$, such that

$$3\left(nn - 1\right) = \frac{ff}{gg}\left(n + 1\right)^2.$$

Solving for $n$ makes

$$n = \frac{ff + 3gg}{3gg - ff},$$

from which it follows that

$$a = d = \frac{12ffgg}{(3gg - ff)^2},$$
$$b = nn + 2 = \frac{3f^4 - 6ffgg + 27g^4}{(3gg - ff)^2} \quad \text{and}$$
$$c = \frac{3f^4 + 6ffgg + 27g^4}{(3gg - ff)^2}.$$

These are certainly rational numbers, but they may not be integers. However, they have a common denominator. Because Euler's problem depends on ratios, we can scale the solution to integers and take

$$a = d = 4ffgg,$$
$$b = f^4 - 2ffgg + 9g^4 \quad \text{and}$$
$$c = f^4 + 2ffgg + 9g^4.$$

From here, we can work backwards to find first *p, q, r* and *s* and then *x, y* and *z*. The problem is solved, in theory, so Euler gives us some examples.

Example 1: $f = 1$ and $g = 1$.

Then $a = d = 4$, $b = 8$ and $c = 12$. These scale down to $a = d = 1$, $b = 2$ and $c = 3$. This makes $p = 5$, $q = 5$, $r = 7$ and $s = 1$, which, in turn, makes $x = 50$, $y = 50$ and $z = 14$. Indeed, the sums and differences are 0, 36, 64 and 100, and they are all squares, as required. Still, with $x = y$, some readers may be dissatisfied and think that Euler threw away all the interesting solutions as he simplified his analysis. So he does another example.

Example 2: Example 2: $f = 2$ and $g = 1$.

Then $a = d = 16$, $b = 17$ and $c = 33$, so $p = 800$, $q = 305$, $r = 817$ and $s = 256$, and we get the solution

$$x = 733025, \quad y = 48800 \text{ and } z = 418304.$$

Indeed

$$x + y = 1105^2, \qquad x - y = 495^2,$$
$$x + z = 1073^2, \qquad x - z = 561^2,$$
$$y + z = 952^2 \text{ and } \qquad y - z = 264^2.$$

Euler does two more examples. The values $f = 3$ and $g = 1$ give the same solution as Example 1, and the values $f = 1$ and $g = 2$ give nine-digit values of *x, y* and *z*, which Euler chooses not to write out explicitly.

Euler seems to be done, but in absolutely typical Euler style, he has more to say. In fact, he has two more things to say. The first he calls a "Note" and the second he calls an "Addition."

In the Note, Euler observes that new solutions can be constructed from old ones. Suppose that *x, y* and *z* solve the problem. Then we get another solution by taking

$$X = \frac{yy + zz - xx}{2}, \qquad Y = \frac{xx + zz - yy}{2} \quad \text{and} \quad Z = \frac{xx + yy - zz}{2}$$

It is immediate to check that the sums are squares, and it is amusing to check that the differences are squares as well.

In the Addition, Euler tells us, "only a few more details are required to solve another problem."

Problem: To find three squares, $xx$, $yy$ and $zz$, such that any two differences are squares.

For any values of $p$ and $q$, if we set $x = pp + qq$ and $y = 2pq$, then $xx - yy = (pp - qq)^2$. Likewise, whatever $r$ and $s$ are, if we set $x = rr + ss$ and $z = 2rs$, then $xx - zz = (rr - ss)^2$. All that remains is to make sure that the two values of $x$ match up and that $yy - zz$ is a square. This amounts to the two conditions

$$pp + qq = rr + ss \text{ and}$$
$$yy - zz = 4(ppqq - rrss).$$

As before, this makes

$$p = ac + bd, \quad q = ad - bc, \quad r = ad + bc \quad \text{and} \quad s = ac - bd.$$

Substitution and factoring shows that for $yy - zz$ to be a square, the product

$$abcd(aa - bb)(dd - cc)$$

must also be a square. We see that this can be done if

$$a = d = nn \pm 1, \quad b = 2mm \mp 1 \quad \text{and} \quad c = nn \mp 2.$$

Indeed, if instead of the integer $n$ we use the rational number $\frac{m}{n}$, we get the following pair of generators:

$$a = d = mm \mp nn, \quad b = 2mm + nn \quad \text{and} \quad c = mm \pm 2nn.$$

Taking simple numbers for $m$ and $n$, Euler gives us a table of 23 pairs of values of $a$, $b$, $c$ and $d$, each of which gives a solution to the new problem, a total of 46 such solutions.

Then he gives us an example starting from the numbers $m = 2$ and $n = 1$, taking the lower signs. This makes $a = 5$, $b = 7$, $c = 5$ and $d = 2$ (where Euler has reversed $c$ and $d$, not that it matters). Then $p = 39$, $q = 25$, $r = 45$ and $s = 11$, whence

$$x = 2146, \quad y = 1950, \quad z = 990$$

or, scaling down by the common divisor 2,

$$x = 1073, \quad y = \quad 975, \quad z = 495.$$

A check shows that the differences of the squares of these values are the squares of 448, 952 and 840.

Finally, Euler notes that if *x, y* and *z* are a solution to the problem, then another solution can be generated by taking

$$\begin{aligned} X &= 2(yy + zz - xx), \\ Y &= 2(xx + zz - yy) \quad \text{and} \\ Z &= 2(xx + yy - zz). \end{aligned}$$

Let us turn now to *why* Euler wrote this paper. As we mentioned above, Euler tells us only that others have worked on the problem, not who they were or why they were interested. Rudolf Feuter, the Editor of the volume of the *Opera omnia* in which this article is reprinted, sheds no light on the question either. We are left to figure it out or shrug it off as something we might never know.

On the other hand, we might let the three sums, $x + y$, $x + z$ and $y + z$, be $a^2$, $b^2$ and $c^2$, and the three differences be $d^2$, $e^2$ and $f^2$, respectively. Note that the values *a, b, c* and *d* have a different meaning than they did above. Then a variety of Pythagorean triples arise by calculations like the following:

$$\begin{aligned} a^2 &= x + y \\ &= (x + z) + (y - z) \\ &= b^2 + f^2, \end{aligned}$$

or

$$\begin{aligned} a^2 &= x + y \\ &= (x - z) + (y + z) \\ &= e^2 + c^2. \end{aligned}$$

Such calculations lead to two more Pythagorean triples, $b^2 = c^2 + d^2$ and $e^2 = d^2 + f^2$. So, we have four right triangles and six edges. That makes us think about a tetrahedron. A few minutes of sketches and puzzling reveals that we are looking for a tetrahedron with integer sides and with right triangles for all four faces. The reader is encouraged to draw some pictures. As you try to imagine such a tetrahedron, you should realize that the side of length *a* is not incident to any right angles and the side of length *d* has two right angles at each end.

Similarly, we can discover the reason behind the problem in Euler's Addition. Let the differences of squares be $a^2 = x^2 - y^2$, $b^2 = x^2 - z^2$ and $c^2 = y^2 - z^2$. This gives three Pythagorean triples, (*y, a, x*), (*z, b, x*) and (*z, c, y*). Again, we have a tetrahedron, one with three right triangles. The triangle that is *not* a right triangle has sides of length *a, b* and *c*. Again, the reader should draw a picture and note that the three right angles do not share a common vertex. Such a tetrahedron would satisfy a different system of Diophantine equations, which we will leave the reader to discover.

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