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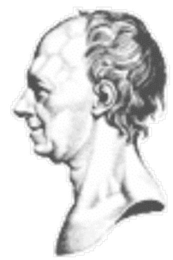

How Euler Did It

by Ed Sandifer

Sums of powers

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Many people have been interested in summing powers of integers. Most readers of this column know, for example, that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2},$$

and many of us even have our favorite proofs. Those of us who have studied or taught mathematical induction lately are likely to recall that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6},$$

and it is obvious that

$$\sum_{i=1}^{k} i^0 = k.$$

Indeed, some days it seems like the main reason mathematical induction exists is so that we can make students prove identities like this. Euler knew all of these identities as well. In fact, he knew them at least up to the eighth exponent:

$$\sum_{i=1}^{k} i^8 = \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 - \frac{7}{15}x^5 + \frac{2}{9}x^3 - \frac{1}{30}x.$$

Jakob Bernoulli [Bernoulli 1713] had given a comprehensive account of sums of powers in his posthumous epic, *Ars conjectandi*, a masterpiece laying out the complete foundations of the theory of discrete probability. Euler surely knew this book well, as Bernoulli had left it to his son, Nicolaus, to edit and publish the volume, and the younger Bernoulli had been a good friend of Euler.

Bernoulli's solution to the problem involved what is still rather advanced mathematics and a special sequence of numbers known as Bernoulli numbers. His work was complete and correct, but it was by no means easy or elementary. Thus, when Euler came on the scene, the problem was not *how* to sum the powers of integers, but how to do it simply, in a way that might be easy to follow and easy to remember. Euler considered this problem late in his life in an article he wrote in 1776, *De singulari ratione differentiandi et integrandi, quae in summis serierum occurrit*, "On a singular means of differentiating and integrating which occurs in the summing of series." [E642]

To Euler, calculus was easy, and notation was something he made up to fit the circumstances. He was also a genius at spotting patterns and doing calculations. When he approached this problem in 1776, he was also blind and aided by assistants who wrote down his words as Euler dictated articles to him. Euler wrote Σx^n to denote the sum of the nth powers of the first x natural numbers. That is,

$$\Sigma x^n = 1^n + 2^n + 3^n + 4^n + \cdots + x^n .$$

He found Σx^n for n from 0 to 8 and started looking for patterns.

$$\begin{aligned}
&\Sigma x^0 = x \\
&\Sigma x^1 = \frac{1}{2}x^2 + \frac{1}{2}x \\
&\Sigma x^2 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x \\
&\Sigma x^3 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 \ + * \\
&\Sigma x^4 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 \ + * - \frac{1}{30}x \\
&\Sigma x^5 = \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 + * - \frac{1}{12}x^2 + * \\
&\Sigma x^6 = \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 \ + * - \frac{1}{6}x^3 \ + * + \frac{1}{42}x \\
&\Sigma x^7 = \frac{1}{8}x^8 + \frac{1}{2}x^7 + \frac{7}{12}x^6 + * - \frac{7}{24}x^4 + * + \frac{1}{12}x^2 + * \\
&\Sigma x^8 = \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 \ + * - \frac{7}{15}x^5 \ + * + \frac{2}{9}x^3 \ + * - \frac{1}{30}x \\
&\text{etc.}
\end{aligned}$$

Here, to help us spot patterns a bit more easily, Euler uses a "*" to indicate that a term of the polynomial is missing. Descartes had used also used a "*" as a placeholder 150 years earlier.

Some patterns are quite obvious. For example, the first term of Σx^n is always $\frac{1}{n+1}x^{n+1}$. That looks suspiciously like an integral. The second term is always $\frac{1}{2}x^n$. Also, once a "*" or a negative term is introduced, it stays in that position in all subsequent formulas. It is not so clear from the given formulas, but the terms continue to alternate between zero and non-zero terms, and the signs of the non-zero terms alternate as well. Readers familiar with the Bernoulli numbers will recall that they share similar properties.

There are two other patterns we might not notice, but once we do notice them, their reason becomes obvious. First, there is never a constant term. This is because whenever $x = 0$ we also have $\Sigma x^n = 0$. Likewise, the sum of the coefficients is always equal to 1 because whenever $x = 1$ we also have $\Sigma x^n = 1$.

That was the easy part. Euler sees some more subtle patterns, which he describes as follows. Start with

$$\Sigma x^0 = x \ ,$$

which, for purposes of exposition, we'll write as

$$\Sigma x^0 = x + 0 \, .$$

Multiply the two terms the right hand side by

$$\frac{1}{2}x \quad \text{and} \quad \frac{1}{1}x \, ,$$

respectively, and add a linear term of the form αx, where α is chosen to make the sum of the coefficients equal to 1, in accordance with the pattern we saw above. In this case, $\alpha = \frac{1}{2}$ and we get the formula for Σx^1, namely

$$\Sigma x^1 = \frac{1}{2}x^2 + \frac{1}{2}x.$$

Now, multiply the *three* terms (we didn't write "+0" this time, but we count it anyway) by

$$\frac{2}{3}x, \quad \frac{2}{2}x \quad \text{and} \quad \frac{2}{1}x \, ,$$

respectively, and again add a linear term of the form αx, where α is chosen to make the sum of the coefficients equal to 1. This time, $\alpha = \frac{1}{6}$ and we get

$$\Sigma x^2 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x \, .$$

In general, Euler has us multiply the terms of Σx^n by

$$\frac{n+1}{n+2}x, \quad \frac{n+1}{n+1}x, \quad \frac{n+1}{n}x, \quad \ldots \quad \frac{n+1}{2}x \quad \text{and} \quad \frac{n+1}{1}x \, ,$$

respectively, and add the appropriate linear term and to get the formula for Σx^{n+1}. Euler notes that the values of α, namely 1, $\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$, 0, $-\frac{1}{30}$, 0, etc., are exactly the Bernoulli numbers. He has seen them several times before, particularly in his evaluation of the zeta function, $\sum_{k=1}^{\infty}\frac{1}{k^n}$, for even values of *n*, and in the Euler-Maclaurin summation formula.

Having observed the pattern, Euler now tries to explain it. He introduces calculus, as suggested by the pattern of the first terms of each sum, and claims that

$$\begin{aligned}
\Sigma\, x^1 &= 1\int \partial x\, \Sigma\, x^0, \\
\Sigma\, x^2 &= 2\int \partial x\, \Sigma\, x^1, \\
\Sigma\, x^3 &= 3\int \partial x\, \Sigma\, x^2, \\
\Sigma\, x^4 &= 4\int \partial x\, \Sigma\, x^3, \\
\Sigma\, x^5 &= 5\int \partial x\, \Sigma\, x^4, \\
&\text{etc.,}
\end{aligned}$$

and, in general,

$$\Sigma x^{n+1} = (n+1)\int \partial x\, \Sigma\, x^n.$$

We can see what Euler is thinking here. He wants to reverse integration and summation and manipulate symbols something like:

$$\begin{aligned}
(n+1)\int \partial x\, \Sigma\, x^n &= (n+1)\sum\int x^n\, dx \\
&= (n+1)\sum\frac{1}{n+1}x^{n+1} \\
&= \Sigma\, x^{n+1}.
\end{aligned}$$

This doesn't quite make sense, even though the editors of the *Nova acta academiae scientiarum Petropolitanae* call it *une demonstration rigoreuse*, "a rigorous proof". Let's see how it works.

We know that $\Sigma\, x^0 = x$. Then making a partial shift to modern notation,

$$\begin{aligned}
1\int \Sigma\, x^0\, dx &= 1\int x\, dx \\
&= \frac{1}{2}x^2 + C,
\end{aligned}$$

where C isn't really a constant, but instead is a linear term, $\frac{1}{2}x$. That didn't quite work.

Continuing to $n = 2$, Euler claims

$$\begin{aligned}\Sigma x^2 &= 2\int \partial x \, \Sigma x^1 \\ &= 2\int \left(\frac{1}{2}x^2 + \frac{1}{2}x\right)dx \\ &= 2\left(\frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 2}x + C\right) \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2C,\end{aligned}$$

where again $2C$ isn't really a constant but has to be $\frac{1}{6}x$. We see that the pattern continues; integration gives most of the next formula, but the appropriate linear term has to be added as well.

In a similar vein, Euler gives us a formula involving differentiation,

$$\Sigma x^n = \frac{\partial \cdot \Sigma x^{n+1}}{(n+1)\partial x}.$$

This almost makes sense in the same, quirky symbolic way that the integral formula did, but it doesn't quite work, either. For the case $n = 1$, for example, this would give

$$\begin{aligned}\frac{1}{2}x^2 + \frac{1}{2}x &= \Sigma x^1 \\ &= \frac{\partial \cdot \Sigma x^2}{2\partial x} \\ &= \frac{1}{2}\frac{d}{dx}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right) \\ &= \frac{1}{2}\left(x^2 + x + \frac{1}{6}\right),\end{aligned}$$

which isn't true because of the constant term in the last line. To make Euler's differential formula correct, we have to throw away the constant term after taking the derivative. So, Euler's paper isn't as rigorous as his editors thought it was.

Euler closes the paper with a number of examples, mostly summing the values of polynomials, but there is an application to the Euler-Maclaurin formula that takes a bit of insight.

In the end, this is a pleasant paper, but we're left to wonder why Euler wrote this paper? Most of the pattern is in Bernoulli's work from 60 years earlier, though Euler is clearer. I speculate that this wasn't really a research paper, but Euler's effort to teach the material to his assistants, who were also, in many ways, his students. To write the paper, Euler's words had to go through his students, and it is likely that he had the students work out the examples at the end of the paper. They would have learned from the experience.

I've learned from the experience, too. The next time I can't remember $\sum_{i=1}^{n} i^2$, I'm going to remember that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$. I'll integrate that, discard the constant and double it to get $\frac{1}{3}n^3 + \frac{1}{2}n^2$. Then, I'll add the appropriate linear term to get

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

If I can remember.

References:

[Bernoulli 1713] Bernoulli, Jakob, *Ars conjectandi*, Basel, Imprensis Thurnisiorum fratrum, 1713. English translation by Edith Dudley Sylla, Johns Hopkins UP, Baltimore, 2006.

[E642] Euler, Leonhard, De singulari ratione differentiandi et integrandi, quae in summis serierum occurrit, *Novi acta academiae scientiarum Petropolitanae* **6** (1788) 1790, pp. 3-5, 79-80. Reprinted in *Opera omnia*, Series I vol. 16a, pp. 122-138. Also available online, along with an English translation by Adam Glover, at EulerArchive.org.

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