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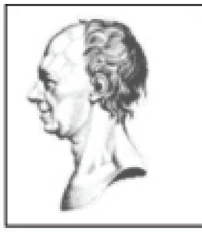
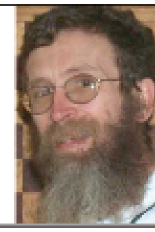


How Euler Did It

by Ed Sandifer



The Moon and the Differential

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Euler's output was split fairly evenly between pure and applied mathematics, the latter including many topics that we would today classify as physics. Most of his papers fall decisively into one category or the other, but it wasn't at all rare for one of his works of applied mathematics to include new techniques or results in analysis. This frequently happened in the Paris Prize competition, for example, where the questions were generally of a practical nature. This month, we'll look at an astronomical paper [E401] that proposes numerical techniques for approximating a body's velocity and acceleration. Remarkably, one of the results in E401 was probably the first step in the development of the calculus of operations and seems to have influenced Lagrange's foundational program for the calculus.

Euler read E401, "A New Method for Comparing Observations of the Moon to the Theory," to the Berlin Academy on February 6, 1766, just a few months before his return to St. Petersburg. Because he quoted astronomical data from the summer of 1765, it's nearly certain that his results date that year. Nevertheless, the paper appeared in the Berlin Academy's volume for 1763, which was only published in 1770. Euler probably had it included in this volume, despite its later composition date, because it's a follow-up to E398, "A New Method for Determining the Perturbations in the Motion of Heavenly Bodies Caused by their Mutual Attraction," which he read to the Academy on July 8, 1762. Because of the long publication delay in the 1763 volume, Euler was able to arrange matters so that E398 was immediately followed by three papers that build upon it results: E399, read on December 18, 1763, which applies the methods E398 to the moon, E400, read on December 4, 1765, which considers the general three body problem, and E401.

Euler begins E401 by summarizing what he had done in E398. Supposing

> "...first that both the position of the body in question, as well as its motion, i.e. its speed and direction, are exactly known for a given epoch; and further, knowing at the same time the accelerations produced by the forces exerted on the body, I showed how we may assign the position and motion of the body from this, not just an instant later, but for a considerable enough time after the first instant."

Euler observes that it would be quite useful to apply this method to the determination of the motion of the moon and the construction of lunar tables. However, he says that he initially despaired of ever being able to measure the moon's velocity with sufficient accuracy, having "neither the fortitude nor the patience to undertake a task of this kind; but...I found a method, by using various observations of the moon, made on several consecutive days, to ascertain for each one the true speed and direction of the moon." That is, he had figured out a numerical method for determining the moon's velocity (and for that matter its acceleration) from a sequence of observations of the moon's position. He describes this in the following proposition.

"**Lemma.** *For the abscissas $\zeta = 0$, $\zeta = 1$, $\zeta = 2$, $\zeta = 3$, $\zeta = 4$, etc., knowing their ordinates p, q, r, s, t, etc. on a curve, it is required to find the differential values both of the first degree $\frac{\mathrm{d}p}{\mathrm{d}\zeta}$, $\frac{\mathrm{d}q}{\mathrm{d}\zeta}$, $\frac{\mathrm{d}r}{\mathrm{d}\zeta}$, $\frac{\mathrm{d}s}{\mathrm{d}\zeta}$, etc., as well as the second degree $\frac{\mathrm{dd}p}{\mathrm{d}\zeta^2}$, $\frac{\mathrm{dd}q}{\mathrm{d}\zeta^2}$, $\frac{\mathrm{dd}r}{\mathrm{d}\zeta^2}$, $\frac{\mathrm{dd}s}{\mathrm{d}\zeta^2}$, etc., taking the differential $\mathrm{d}\zeta$ to be constant.*"

Euler's notation may seem a little strange to modern readers. The letters p, q, r, etc., don't represent different functions, only different values of a given function, which we might call $z = f(\zeta)$. Therefore we would write something more like

$$\left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=0}, \quad \left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=1}, \quad \left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=2}, \quad \cdots$$

where Euler has written $\frac{\mathrm{d}p}{\mathrm{d}\zeta}$, $\frac{\mathrm{d}q}{\mathrm{d}\zeta}$, $\frac{\mathrm{d}r}{\mathrm{d}\zeta}$, etc. It's even clearer in Lagrange's derivative notation, not yet invented in 1765: Euler is simply trying to find $f'(0)$, $f'(1)$, $f'(2)$, $f'(3)$, ... as well as $f''(0)$, $f''(1)$, $f''(2)$, $f''(3)$, ...

Euler's solution to the problem posed in this lemma begins with some notation. He defines z as we have done, then he sets $q-p = \Delta p$, $r-2q+p =$

$\Delta^2 p$, $s - 3r + 3q - p = \Delta^3 p$, etc. If we let $z_n = f(n)$, then these are the forward differences Δz_0, $\Delta^2 z_0$, $\Delta^3 z_0$, ... "Given this," says Euler, "we know that we have:

$$\begin{aligned} z &= p + \Delta p.\frac{\zeta}{1} + \Delta^2 p.\frac{\zeta(\zeta-1)}{1.2} + \Delta^3 p.\frac{\zeta(\zeta-1)(\zeta-2)}{1.2.3} \text{ etc., or} \qquad (1) \\ &= p + \Delta p\,\zeta + \frac{1}{2}\Delta^2 p(\zeta\zeta - \zeta) + \frac{1}{6}\Delta^3 p(\zeta^3 - 3\zeta^2 + 2\zeta) \\ &\quad + \frac{1}{24}\Delta^4 p(\zeta^4 - 6\zeta^3 + 11\zeta^2 - 6\zeta) \\ &\quad + \frac{1}{120}\Delta^5 p(\zeta^5 - 10\zeta^4 + 35\zeta^3 - 50\zeta^2 + 24\zeta) \\ &\qquad \text{etc."} \end{aligned}$$

Equation (1) is sometimes called Newton's Forward Difference Formula. Many of us have come across it in a numerical methods course, see e.g. [Burden 2001, p. 127], where the name is usually applied to an interpolating polynomial, rather than an infinite series. If $\{x_n\}$ is a sequence with constant differences $h = \Delta x_i$, then

$$p_n(t) = \sum_{k=0}^{n} \binom{t}{k} \Delta^k f(x_0) \qquad (2)$$

is the unique polynomial of degree $\leq n$ with the property that $p_n(k) = f(x_k)$ for $k = 0, 1, 2, \ldots, n$. When $x \in [x_0, x_n]$ and $x = x_0 + th$, then $f(x) \approx p_n(t)$. In Euler's application, $x_0 = 0$ and $h = 1$, so that t in equation (2) is just his variable ζ. Furthermore, if $f(x)$ is a well-behaved function, then under certain conditions we will have $p_n(t) \to f(x)$ as $n \to \infty$, which more or less justifies Euler's claim in equation (1).

Next, Euler differentiates equation (1) to get

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\zeta} &= \Delta p + \frac{1}{2}\Delta^2 p\,(2\zeta - 1) + \frac{1}{6}\Delta^3 p\,(3\zeta\zeta - 6\zeta + 2) \\ &\quad + \frac{1}{24}\Delta^4 p\,(4\zeta^3 - 18\zeta^2 + 22\zeta - 6) \\ &\quad + \frac{1}{120}\Delta^5 p\,(5\zeta^4 - 40\zeta^3 + 105\zeta^2 - 100\zeta + 24) \\ &\qquad \text{etc.,} \end{aligned}$$

an expression that is sometimes called Markoff's Formula [MathWorld]. Euler then differentiates a second time to find a similar expression for $\frac{\mathrm{dd}z}{\mathrm{d}\zeta^2}$.

Finally, Euler substitutes $\zeta = 0$, $\zeta = 1$, $\zeta = 2$, etc., to find the first and second order differential quantities he set out in the statement of the lemma. The first of these is

$$\frac{\mathrm{d}p}{\mathrm{d}\zeta} = \Delta p - \frac{1}{2}\Delta^2 p + \frac{1}{3}\Delta^3 p - \frac{1}{4}\Delta^4 p + \frac{1}{5}\Delta^5 p - \text{ etc.} \tag{3}$$

For good measure, he also derives formulas for the third and fourth derivatives.

Now let's skip ahead about half a century, to 1814. In that year, François-Joseph Servois (1768-1847) published a paper called "Essay on a new method of exposition for the differential calculus" [Servois 1814a] in *Annales de mathématiques pures et appliqués*. Often called "Gergonne's *Annales*" after its editor, this was the first journal ever to be devoted entirely to mathematics. Servois' paper contained the following remarkable *definition*:

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \ldots = dz, \tag{4}$$

for an arbitrary function z. "This is the complete definition of a new function of z," says Servois, "polynomial or even *infinitinomial*,[1] in general, which I call the *differential*."

In this paper, Servois was grappling with the foundational problem of the calculus. At the beginning of the 19th century, there were three competing foundational schools on the European Continent: those who thought that differentials were acceptable or at least could be made suitably rigorous, those who wanted to base calculus on the limit – still an informal notion at that time – and a third group who, following Lagrange (1736-1813), defined derivatives not via limits, but rather through the coefficients of a function's power series expansion. Servois was a disciple of Lagrange and his paper was full of formal series manipulations, including a derivation of the expansion (1). Although he was reasonably sympathetic to the limit approach, as he demonstrated in a philosophical essay that followed immediately in the pages of Gergonne's *Annales* [Servois 1814b], he wanted to banish the infinitely small from mathematics. However, he recognized that the use of differentials was deeply ingrained in mathematical practice, so he sought to explain them here through formal operations rather than through an appeal to some vague notion of infinitely small quantities.

[1] Servois coined this term (*infinitinôme*) here. Although this word never caught on, he also introduced the mathematical terms "distributive" and "commutative" in this paper.

Servois' formula (4) gives dz in terms of a constant increment in the independent variable, which he denoted by a Greek letter such as α. If we call it $d\zeta$ instead and formally divide it through both sides of (4), we get Euler's formula (3). It's extremely unlikely that Servois gleaned his definition of the differential directly from E401, although his publication record make it quite clear that he was very familiar with Euler's works. Rather, the line from formula (3) to definition (4) passes through the works of Lagrange, Arbogast (1759-1803) and Jacques Français (1775-1833). Arbogast and Français established the calculus of operations, in which operators were abstracted from the functions to which they were applied, so that a formula like (3) could be re-written as

$$D = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \dots$$

where D is the derivative operator. Because the right hand side has the form of the power series for the natural log, Français wrote Euler's formula as

$$D = \ln(1 + \Delta) \tag{5}$$

and Servois said that the differential is the logarithm of what he called the "varied state," i.e. the forward increment operator that maps z to $z + \Delta z$.

Ivor Grattan-Guinness traces this evolution in [Grattan-Guinness 1990, pp. 161-163, 211-219]. The next step after E401 was taken by Lagrange, who succeeded Euler at the Berlin Academy. In [Lagrange 1774], he not only generalized Euler's formula (3) to the multivariable case, but he produced its dual, by showing that

$$\Delta z = f(\zeta + h) - f(\zeta) = e^{h\frac{\mathrm{d}z}{\mathrm{d}\zeta}} - 1$$

where h, the increment for the difference operator, was taken by Euler to be 1 in E401. Français could derive the corresponding $\Delta = e^D - 1$ by formally solving relation (5) for Δ, but Lagrange derived his result from the Taylor series expansion of the function z.

We should note carefully that none of these later theoretical consequences of E401 were foreshadowed in any way by Euler himself. He must have observed the elegance of the expression (3) of the derivative in terms of differences and noticed the analogy with the logarithm series. However, in this paper it was just a means to a practical end: the construction of accurate lunar tables and, by extension, progress on the Longitude Problem.

To illustrate the use of his newly discovered tool, Euler gathered astronomical data from the lunar tables of Jérôme LaLande (1732-1807) for Paris on six consecutive days, July 31 through August 5 of 1765. His coordinate system takes the center of the earth as the origin, with the plane of the ecliptic as the xy-plane and the positive x-axis pointing in the direction of the vernal equinox. With units chosen so that the mean distance from the earth to the sun is 100,000, the position (x, y, z) of the moon at noon, ζ days after August 1, 1765, is approximately given by the quadratic formulas

$$\begin{array}{rcrcrcr} x & = & 166.970 & + & 36.090\zeta & - & 5.104\zeta\zeta \\ y & = & -184.039 & + & 40.316\zeta & + & 5.618\zeta\zeta \\ z & = & -9.7545 & + & 5.1982\zeta & + & 0.3020\zeta\zeta \end{array}$$

As another consequence of his methods, Euler calculates the ratio of the sun's mass to that of the earth to be 309,108, assuming solar parallax to be $9''$ of arc. This compares reasonably well with the currently accepted figure of 332,830 and would have been much closer had he used the currently accepted value of 8.794 seconds of arc for solar parallax.

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