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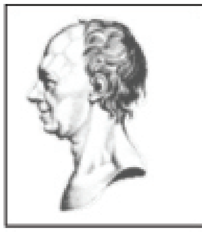

How Euler Did It

by Ed Sandifer

Did Euler Prove Cramer's Rule?

November 2009 – A Guest Column by Rob Bradley

The 200th anniversary of Euler's death took place in September 1983. The milestone was marked in a reasonable number of publications, although fewer than the number that celebrated the tercentenary of his birth two years ago. The MAA devoted an entire issue of the *Mathematics Magazine* to Euler's life and work [Vol. 56, no. 4, Nov. 1983] and there were at least two edited volumes of essays published to mark the event.

Among the many essays included in [Burckhardt 1983] is a piece by Pierre Speziali on Euler and Gabriel Cramer (1704-1752), the same Cramer whose name is attached to the famous rule for solving linear systems. Most of Speziali's article is a survey of the correspondence between Euler and Cramer. The Euler-Cramer correspondence will soon be readily available, because it will be included in the forthcoming Volume 7 of Series IVA of Euler's *Opera Omnia*, scheduled to be published in 2010. Siegfried Bodenmann, a co-editor of IVA.7, discusses the correspondence as an important example of 18th scientific correspondence in his chapter in [Henry 2007].

The correspondence consists of 19 letters in perfect alternation. The first one was a brief letter from Euler, written in 1743. Its contents and tone make it clear that there had previously been no direct contact between the two men. The final letter was written by Euler in late 1751, just a few weeks before Cramer's death. However, the 1975 catalog of Euler's correspondence lists only 17 of these letters. One of the two missing documents was Cramer's final letter to Euler. Although its whereabouts remain a mystery, its contents were known to Speziali and will be included in the *Opera Omnia*, because Cramer's draft survives in the archives of the public library in Geneva, where Cramer lived and taught. The other missing letter was Euler's third to Cramer. It was written at some point between Cramer's letters of September 30 and November 11, 1744, and was entirely unknown to Speziali in 1983.

Speziali pays particular attention to the letter of November 11, 1744, in which Cramer gave Euler a complete description of his rule for solving systems of linear equations. This is noteworthy, because Cramer's Rule would not appear in print until six years later, where it was an appendix in his very influential book *Introduction to the analysis of algebraic curves* [Cramer 1750]. What's even more interesting is that the passage in Cramer's letter is virtually identical, word for word, to a three-page passage in the *Introduction* [Cramer 1750, pp. 657-659].

Speziali further notes that immediately before describing his famous rule, Cramer says to Euler "your remark cannot but strike me as quite correct, because it agrees entirely with my own thoughts on the subject." Speziali goes on to say that it is "very regrettable that Euler's letter is lost because – who knows? – it might have revealed to us a rule similar to Cramer's, or an original idea that had inspired the latter." That is, he speculates that Euler might have discovered Cramer's Rule independently of Cramer, or that he might even deserve priority for it, by communicating a result in his lost letter that led Cramer to the discovery of his rule. If only that third letter to Cramer hadn't been lost ...

The lost letter became known to Euler scholars at the meeting of the Euler Society in August 2003. At some point in the 20th century, it found its way into the private collection of Bern Dibner (1897-1988). Dibner was an engineer, entrepreneur and philanthropist, as well as a historian of science. Over the course of his long life, he amassed an impressive private collection of rare books, manuscripts and letters. He donated about a quarter of this collection to the Smithsonian in 1974 and Euler's missing letter of October 20, 1744, was part of that gift. Mary Lynn Doan, professor of mathematics at Victor Valley Community College, had contacted the Dibner Library of the Smithsonian Institution in the summer of 2003 and had learned that they have a small collection of documents by Leonhard Euler [Euler Papers]. She visited the Library on her way to the Euler Society's meeting that summer and brought a photocopy of the letter with her. I was able to identify the addressee as Cramer and shortly afterwards I brought the letter to the attention of Andreas Kleinert, co-editor of the forthcoming volume IVA.7 of the *Opera Omnia*. Thanks to Mary Lynn and the excellent archivists at the Smithsonian, Euler's *Opera Omnia* will now include the complete correspondence with Cramer.

Does Euler's letter, now cataloged as R.461a in the *Opera Omnia*, show that he knew Cramer's Rule before Cramer did? Not in the least. In fact, one of the things it tells us is that when Cramer wrote in November that Euler's remark "agrees entirely with my own thoughts," Euler had actually been talking about *Cramer's Paradox*, not Cramer's Rule.

Cramer's Paradox was the subject of an earlier *How Euler Did It* column [Sandifer 2004]. The simplest case of Cramer's Paradox involves two curves of the third degree. On one hand, two curves of degree m and n can intersect in as many as mn points. This theorem, named after Etienne Bézout (1730-1783), implies that two cubic curves may intersect in nine places. On the other hand, an equation of degree n has $\frac{(n+1)(n+2)}{2} = \frac{n^2+3n}{2} + 1$ coefficients. Because an equation can be multiplied by an arbitrary constant without affecting its graph, $\frac{n^2+3n}{2}$ points typically determine a curve of order n. Thus nine points in general position *uniquely* determine a cubic curve, and yet *two* such curves can typically intersect in nine points. In R.461a, Euler proposed the following resolution of the paradox:

> "I say, then, that although it is indeed true that a line of order n be determined by $\frac{nn+3n}{2}$ points, this rule is nevertheless subject to certain exceptions. ...it may happen that such a number of equations, which we draw from the same number of given points, is not sufficient for this effect: this is evident, when two or several of these equations become identical. ...I conceive therefore, that this inconvenience will take place when the nine points, which ought to determine a line of the 3rd order, are disposed such that two curved lines of this order may be drawn through them. In this case, the nine given points, because they include[1] two identical equations, are worth but 8,...From this, one easily understands that if the nine points, from which one ought to draw a line of the third order, are at the same time the intersections of two curved lines of this order, then, after having completed all of the calculations, there must remain in the general equation for this order an undetermined coefficient, and beginning from this case not only two, but an infinity of lines of the 3rd order may be drawn from the same nine points."

[1] Euler used the French verb *renfermer*, which means both to include and to hide. This is particularly appropriate, because the "two identical equations" can only be derived through considerable calculation.

Euler and Cramer agreed that this was the correct resolution of the paradox. Euler described it in his article [E147], which is discussed at length in [Sandifer 2004]. Cramer also gave his account in [Cramer 1750]. Agnes Scott judged that "Euler's resolution of the paradox ... agrees with that of Cramer, and goes just as far, or a little bit further" [Scott 1898, p. 263] , but it was Cramer's name that became attached to it.

In modern terms, the question of determining the equation of a cubic curve reduces to solving a system of 9 homogeneous linear equations in 10 unknowns. The unknowns are the coefficients of the general equation of the third degree

$$\alpha x^3 + \beta x^2 y + \gamma x y^2 2 + \delta y^3 + \varepsilon x^2 + \zeta x y + \eta y^2 + \theta x + \iota y + \kappa = 0. \quad (1)$$

Every time we plug the coordinates (x_0, y_0) of a particular point into equation (1), we have a homogeneous linear equation in the 10 unknowns. If the nine given points lead to a system of rank 9, then there is a unique solution, up to scalar multiplicity. However, when the nine points in question are not generic, but happen to be the points of intersection of two cubic curves, then the rank of the resulting linear system is no greater than 8. Cramer and Euler did not have these definitions and concepts of modern linear algebra at their disposal, but they certainly understood that the question reduced to the solution of linear systems, and this is why Cramer described his famous Rule to Euler in his reply of November 1744.

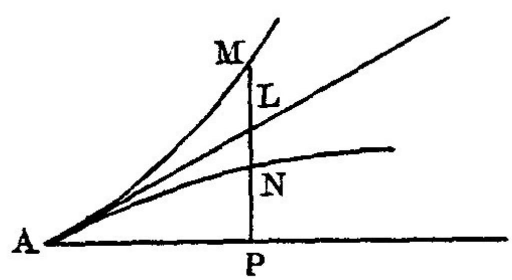

Figure 1: Cusp of the First Kind

Euler's lost letter contains more than just a discussion of Cramer's Paradox. In fact, it contained something of a bombshell: Euler announced that he had just discovered a simple curve that exhibited something called a *cusp of the second kind* or a *ramphoid* cusp. A *cusp of first kind* or *keratoid* cusp is illustrated in figure (1) – the figure is from [E169]. The curve consists of two branches, AM and AN, with a cusp at A and a common tangent AL at that point. The two branches have opposite concavity with respect to the common tangent. A simple example of such a point is the origin in the graph of $y = x^{2/3}$, where the common tangent is the y-axis. We note that this is a cubic curve, because the equation can be re-written in the form of equation (1) as $y^3 - x^2 = 0$.

Figure (2), also from [E169], illustrates a *cusp of second kind.* Once again, there are of two branches, a cusp at A, and a common tangent at that point. This time, however, the two branches have the same concavity with respect to the tangent. L'Hôpital (1661-1704) is responsible for defining these two types of cusps. In 1740, Jean-Paul de Gua de Malves (1713-1785) published a proof that no algebraic curve could have a cusp of the second kind in [Gua de Malves 1740].

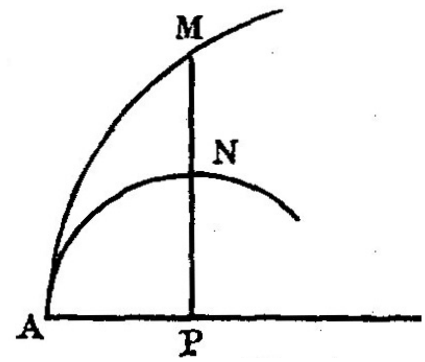

Figure 2: Cusp of the Second Kind

Euler was familiar with Gua de Malves' work and had initially accepted his result, but in 1744 he discovered that there was a subtle flaw in the supposed proof. In R.461a, he wrote to Cramer that "even in the fourth order there is a curved line of this kind, whose equation is $y^4 - 2xy^2 + xx = x^3 + 4yxx$, which simplifies to $y = \sqrt{x} \pm \sqrt[4]{x^3}$."

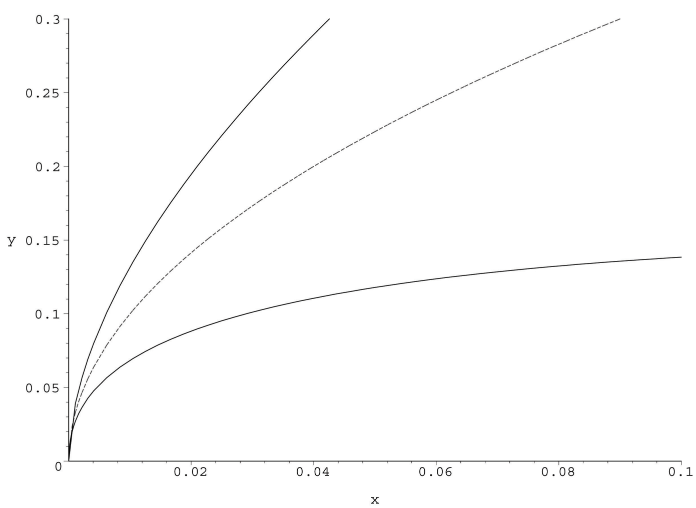

Figure 3: Euler's Example

It's much easier to graph Euler's example in the form $y = \sqrt{x} \pm x^{3/4}$ than as an equation of the fourth degree. In figure (3), the curve is illustrated as a solid line, with the dotted graph of of $y = \sqrt{x}$ added for reference. The branch $y = \sqrt{x} + x^{3/4}$ lies above the square root and the other branch $y = \sqrt{x} - x^{3/4}$ lies below. This picture makes it clear why Euler referred to this curve as “a bird’s beak” in [E169].

What may be less clear is that the two forms of the equation given by Euler are equivalent. To see this, begin by adding $4xy^2$ to both sides of $y^4 - 2xy^2 + xx = x^3 + 4yxx$. We then have

$$\begin{aligned} y^4 + 2xy^2 + x^2 &= x(x^2 + 4xy + 4y^2), \quad \text{or} \\ (y^2 + x)^2 &= x(x + 2y)^2. \end{aligned}$$

Let’s observe that we must have $x \geq 0$, because the two squares are non-negative. Taking square roots, we have

$$\begin{aligned} y^2 + x &= \pm\sqrt{x}(x + 2y), \quad \text{or} \qquad (2) \\ y^2 \mp 2\sqrt{x}y + x &= \pm x^{3/2}. \qquad (3) \end{aligned}$$

The left side of equation (3) is a perfect square, so we reject the case $-x^{3/2}$ on the right side. That means that equation (2) reduces to $y^2 + x = \sqrt{x}(x + 2y)$. Subtracting the term $2\sqrt{x}y$ from both sides, we have

$$\begin{aligned} (y - \sqrt{x})^2 &= x^{3/2} \quad \text{and so} \\ y - \sqrt{x} &= \pm x^{3/4}, \quad \text{or} \\ y &= \sqrt{x} \pm x^{3/4}. \end{aligned}$$

Euler wrote R.461a shortly after completing his *Introductio in analysin infinitorum.* The second volume of the *Introductio* [E102] is a very thorough treatment of analytic geometry, including a classification of cubic and quartic curves and their equations. [Cramer 1750] dealt with many of the same topics. In the Euler-Cramer correspondence, we have the opportunity to see two giants of the theory of equations in a free exchange of ideas.

Letters in Euler’s hand were prized by collectors in the 19th and 20th century. Because of this, there are quite a few more letters to Euler in the *Opera Omnia* than from him. Every so often, missing letters like R.461a resurface and add to our knowledge of Euler’s achievements. It was tantalizing to think that Euler might have scooped Cramer on his Rule, but the real story is no less captivating.

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